

CONSTRUCTING CHARACTERS OF SYLOW p -SUBGROUPS OF FINITE CHEVALLEY GROUPS

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ABSTRACT. Let q be a power of a prime p , let G be a finite Chevalley group over \mathbb{F}_q and let U be a Sylow p -subgroup of G ; we assume that p is not a very bad prime for G . We explain a procedure of reduction of irreducible complex characters of U , which leads to an algorithm whose goal is to obtain a parametrization of the irreducible characters of U along with a means to construct these characters as induced characters. A focus in this paper is determining the parametrization when G is of type F_4 , where we observe that the parametrization is “uniform” over good primes $p > 3$, but differs for the bad prime $p = 3$. We also explain how it has been applied for all groups of rank 4 or less.

1. INTRODUCTION

Let q be a power of a prime p , and let G be a finite Chevalley group over \mathbb{F}_q and let U be a Sylow p -subgroup of G . We assume that p is not a very bad prime for G ; recall that this means that $p > 2$ if G is of type B_r , C_r or F_4 , and $p > 3$ if G is of type G_2 .

We study the representation theory of U with the aim of determining a parametrization of the irreducible characters of U and a means to construct them as induced characters of linear characters of certain subgroups. Our principal tool for achieving this is a method of successively reducing characters to smaller subquotients of U , which leads to an algorithm whose goal is to determine the irreducible characters of U . An outline of this algorithm is given below and explained more fully in Section 3.

A focus of this paper is to obtain the parametrization in case G is of type F_4 , as stated in the following theorem.

Theorem 1.1. *Let q be a power of an odd prime p and let G be a finite Chevalley group over \mathbb{F}_q of type F_4 . The irreducible characters of U are completely parameterized in Table 7. Moreover, each character can be obtained as an induced character of a linear character of a certain subgroup that can be determined from the information in Table 7.*

As explained later in the introduction, the parametrization is “uniform” over all primes $p > 3$. However, we observe significant differences in the parametrization for the bad prime $p = 3$. These differences shed light on why the prime $p = 3$ is bad for G of type F_4 . In particular, we observe that for $p > 3$ all characters have degree q^d for some $d \in \mathbb{Z}_{\geq 0}$, whereas for $p = 3$ there are characters of degree $q^4/3$. We note that similar behaviour for certain characters of U when G is of type E_6 (for $p = 3$) or E_8 (for $p = 5$) has previously been observed in [LM2].

We have also used our algorithm to determine a parametrization of the irreducible characters of U for classical types up to rank 4. We emphasise that our algorithm gives a construction of each character as an induced character from a character of a certain subgroup of U , which gives a means to calculate the values of these characters, see Theorem 3.8.

In fact for G of rank 4 or less, we obtain that each irreducible character of U can be obtained by inducing a linear character. In addition, we remark that our labelling of the irreducible characters is amenable to the action of a maximal torus and also the field automorphisms; thus it would be straightforward to determine these actions explicitly.

The methods in this paper develop those used by Himstedt and the second and third authors in [HLM1] and [HLM2], and make significant further progress. A full parametrization of the irreducible characters of U for G of type D_4 and every prime p is given in [HLM1]. The so called single root minimal degree almost faithful irreducible characters are parameterized for every type and rank when p is not a very bad prime for G in [HLM2].

The approach used in those papers and this paper is built on partitioning the irreducible characters of U in terms of the root subgroups that lie in their centre, but not in their kernel. Consequently, there are similarities to the theory of supercharacters, which were first studied for the case G is of type A by André, see for example [An]. This theory was fully developed by Diaconis and Isaacs in [DI]. Subsequently it was applied to the characters of U for G of types B, C and D by André and Neto in [AN].

Another approach to the character theory of U is via the Kirillov orbit method, which is applicable for p greater than the Coxeter number of G . In [GMR2], Mosch, Röhrle and the first author explain an algorithm for parameterizing the coadjoint orbits of U , which was applied for G of rank at most 8, except E_8 ; through the Kirillov orbit method this leads to a parametrization of the irreducible characters of U . This was preceded by an algorithm to determine the conjugacy classes of U , see [GMR1].

We note that a reduction procedure for algebra groups similar to ours was given by Evseev in [Ev] and builds on work of Isaacs in [Is2]. For $G = \mathrm{SL}_n(q)$, this led to a parametrization of the irreducible characters of U for $n \leq 13$. Also recently Pak and Soffer have determined the coadjoint orbits of U for $G = \mathrm{SL}_n(q)$ and $n \leq 16$, see [PS]. The situation for G not of type A turns out to be more complicated and we comment more on this below.

There has been considerable other interest in the character theory and conjugacy classes of U . We refer the reader to [LM1] or the introduction to [GMR2] for more information.

Motivation for this work lies in determining generic character tables for U , as has been done for G of type D_4 in [GLM]. This has a view towards applications to the modular character theory of G in nondefining characteristic; in particular, to determining decomposition numbers; see for example [Hi], [HH] and [HN] for applications of the character theory of parabolic subgroups to the modular representation theory of G in certain low rank cases.

We move on to give an outline of our algorithm to parameterize the irreducible characters of U ; we have omitted some details here and a full explanation is given in Section 3. In order to give this outline we require some more notation. We write Φ^+ for the system of positive roots determined by U , and for $\alpha \in \Phi^+$ we denote the corresponding root subgroup by X_α .

In the algorithm we consider certain subquotients of U , which we refer to as quattern groups. A *pattern subgroup* of U is a subgroup that is a product of root subgroups, and a *quattern group* is a quotient of a pattern subgroup by a normal pattern subgroup, we refer to §2.3 for a precise definition. A quattern group is determined by a subset \mathcal{S} of Φ^+ and denoted by $X_{\mathcal{S}}$. Given a subset \mathcal{Z} of $\{\alpha \in \mathcal{S} \mid X_\alpha \subseteq Z(X_{\mathcal{S}})\}$, we define

$$\mathrm{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} = \{\chi \in \mathrm{Irr}(X_{\mathcal{S}}) \mid X_\alpha \not\subseteq \ker \chi \text{ for all } \alpha \in \mathcal{Z}\}.$$

At each stage of the algorithm, we are considering a pair $(\mathcal{S}, \mathcal{Z})$ as above. We attempt to apply one of two possible types of reductions to reduce $(\mathcal{S}, \mathcal{Z})$ to one or two pairs such

that the irreducible characters in $\text{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}$ are in bijection with those irreducible characters corresponding to the pairs we have obtained in the reduction.

The first reduction is based on the elementary but powerful character theoretic result [HLM2, Lemma 2.1], which is referred to as the reduction lemma. In Lemma 3.1, we state and prove a specific version of this lemma, which is the basis of the reduction. This lemma shows that under certain conditions (which are straightforward to check) we can replace $(\mathcal{S}, \mathcal{Z})$ with $(\mathcal{S}', \mathcal{Z})$, where \mathcal{S}' contains two fewer roots than \mathcal{S} , and we have a bijection between $\text{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}$ and $\text{Irr}(X_{\mathcal{S}'})_{\mathcal{Z}}$.

The second reduction is more elementary and used when it is not possible to apply the first reduction. For this we choose a root α such that $\alpha \notin \mathcal{Z}$, but $X_{\alpha} \subseteq Z(X_{\mathcal{S}})$. Then $(\mathcal{S}, \mathcal{Z})$ is replaced with the two pairs $(\mathcal{S} \setminus \{\alpha\}, \mathcal{Z})$ and $(\mathcal{S}, \mathcal{Z} \cup \{\alpha\})$. The justification of this reduction is that $\text{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}$ can be partitioned into the characters in which X_{α} is contained in the kernel, namely $\text{Irr}(X_{\mathcal{S} \setminus \{\alpha\}})_{\mathcal{Z}}$, and the characters in which X_{α} is not contained in the kernel, namely $\text{Irr}(X_{\mathcal{S}})_{\mathcal{Z} \cup \{\alpha\}}$.

We first partition the characters in terms of the root subgroups that lie in their kernel, and then apply the reductions to each part of this partition. After we have successively applied these reductions as many times as possible, we are left with a set $\{(\mathcal{S}_1, \mathcal{Z}_1), \dots, (\mathcal{S}_m, \mathcal{Z}_m)\}$ for some $m \in \mathbb{Z}_{\geq 1}$ such that $\text{Irr}(U)$ is in bijection with the disjoint union

$$\bigsqcup_{i=1}^m \text{Irr}(X_{\mathcal{S}_i})_{\mathcal{Z}_i}.$$

We refer to the pairs $(\mathcal{S}_i, \mathcal{Z}_i)$ as *cores*. In many cases we have that $X_{\mathcal{S}_i}$ is abelian in which case it is trivial to determine $\text{Irr}(X_{\mathcal{S}_i})_{\mathcal{Z}_i}$. The more interesting cases are when $X_{\mathcal{S}_i}$ is not abelian, we refer to these as *nonabelian cores*, where there is still some work required to determine $\text{Irr}(X_{\mathcal{S}_i})_{\mathcal{Z}_i}$.

As proved in Theorem 3.8, the irreducible characters of U corresponding to $\text{Irr}(X_{\mathcal{S}_i})_{\mathcal{Z}_i}$ are actually obtained from irreducible characters of $X_{\mathcal{S}_i}$ by first inflating to a certain pattern subgroup of U and then inducing to U . In particular, this gives a method to construct the characters and, therefore, calculate the values of these characters.

The algorithm has been implemented in the computer algebra system GAP3 [GAP3] using the CHEVIE package [CHEVIE]. For G of rank 4 or less, we have used this and an analysis of the nonabelian cores obtained to determine a parametrization of $\text{Irr}(U)$. The results of the calculation are presented in the appendix for G of types B_4 , C_4 and F_4 .

For the case where G is of type F_4 , we obtain six nonabelian cores. These families of characters show the most interesting behavior. For three of these families the parametrization of $\text{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}$ is significantly different when $p > 3$ and $p = 3$; correspondingly, we get a different expression for the size of $\text{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}$ as a polynomial in q . For $p = 3$, we obtain irreducible characters of degree $q^4/3$, whereas for $p > 3$, we obtain that the degree of an irreducible character is always a power of q .

As mentioned above for G of type A, a similar algorithm is given by Evseev in [Ev], which works in the framework of algebra groups. This allows the algorithm to work with more general subgroups of U ; there is not a natural analogue of algebra groups in general types. A parametrization of the irreducible characters of U for G or type A up to rank 12 is achieved in [Ev]. Indeed up to rank 11 there are no nonabelian cores (when working in the framework of algebra groups).

The lack of an analogue of the more general notion of algebra groups outside type A leading to nonabelian cores in low rank is in our opinion the main reason why the problem outside of type A is more complex. To deal with G of higher rank, a more systematic procedure for dealing with nonabelian cores is necessary, and is a direction for future research. This should be based on our analysis of nonabelian cores in Section 4.

Another direction for future work is to construct generic character tables of U for G of type F_4 , the case of D_4 given in [GLM] serves as a model of how to do this.

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2. PRELIMINARIES

2.1. Background on characters of finite groups. Let G be a finite group, and let H be a subgroup of G . We denote by $Z(G)$ the centre of G , and by $\text{Irr}(G)$ the set of all irreducible characters of G . We write 1_G for the trivial character of G . For a character $\eta \in \text{Irr}(H)$, we write $\eta^G = \text{Ind}_H^G \eta$ for the character of G induced from η , and we denote

$$\text{Irr}(G \mid \eta) = \{\chi \in \text{Irr}(G) \mid \langle \chi, \eta^G \rangle \neq 0\}.$$

For a character $\chi \in \text{Irr}(G)$, we denote

$$\ker(\chi) = \{g \in G \mid \chi(g) = \chi(1)\} \quad \text{and} \quad Z(\chi) = \{g \in G \mid |\chi(g)| = \chi(1)\}.$$

Let N be a normal subgroup of G . We have an inflation map from $\text{Irr}(G/N)$ to $\text{Irr}(G)$ which takes $\chi \in \text{Irr}(G/N)$ to $\tilde{\chi} = \text{Inf}_{G/N}^G \chi \in \text{Irr}(G)$, where $\tilde{\chi}(g) = \text{Inf}_{G/N}^G \chi(g) = \chi(gN)$ for $g \in G$. Given $g \in G$, $x \in N$ and $\psi \in \text{Irr}(N)$, we write x^g for $g^{-1}xg$ and we write ${}^g\psi : N \rightarrow \mathbb{C}$ for the character defined by ${}^g\psi(x) = \psi(x^g)$.

For ease of reference later we recall the following elementary commutativity property of induction and inflation. For $\psi \in \text{Irr}(H/N)$ where $N \leq H \leq G$ and $N \trianglelefteq G$, we have

$$\text{Inf}_{G/N}^G \text{Ind}_{H/N}^{G/N} \psi = \text{Ind}_H^G \text{Inf}_{H/N}^H \psi. \quad (2.1)$$

We next explain an elementary result, which we use in the sequel. Let Z and T be subgroups of $Z(G)$ such that $Z \cap T = 1$. We can identify Z with a subgroup of G/T . Let $\lambda \in \text{Irr}(Z)$ and let $\tilde{\lambda}$ denote its inflation to ZT . Then it is straightforward to show that we have a bijection $\text{Irr}(G \mid \tilde{\lambda}) \longleftrightarrow \text{Irr}(G/T \mid \lambda)$.

The next lemma is key for our algorithm, it was proved in [HLM2, Lemma 2.1] and we refer to it as the reduction lemma. We note that a similar result in the context of algebra groups was previously proved by Evseev in [Ev, Lemma 2.1].

Lemma 2.1 (Reduction lemma). *Let G be a finite group, let $H \leq G$ and let X be a transversal of H in G . Let Y and Z be subgroups of H , and $\lambda \in \text{Irr}(Z)$. Suppose that*

- (i) $Z \subseteq Z(G)$,
- (ii) $Y \trianglelefteq H$,
- (iii) $Z \cap Y = 1$,

(iv) $ZY \trianglelefteq G$,

(v) for the inflation $\tilde{\lambda} \in \text{Irr}(ZY)$ of λ , we have that ${}^x\tilde{\lambda} \neq {}^y\tilde{\lambda}$ for all $x, y \in X$ with $x \neq y$.

Then we have a bijection

$$\text{Irr}(H/Y \mid \lambda) \rightarrow \text{Irr}(G \mid \lambda) \cap \text{Irr}(G \mid 1_Y)$$

$$\chi \mapsto \text{Ind}_H^G \text{Inf}_{H/Y}^H \chi.$$

Moreover, if $|X| = |Y|$, then $\text{Irr}(G \mid \lambda) \cap \text{Irr}(G \mid 1_Y) = \text{Irr}(G \mid \lambda)$.

Let p be a prime, and let $q = p^e$ for $e \in \mathbb{Z}_{\geq 1}$. We let \mathbb{F}_q be the finite field with q elements. Denote by $\text{Tr} : \mathbb{F}_q \rightarrow \mathbb{F}_p$ the trace map, and define $\phi : \mathbb{F}_q \rightarrow \mathbb{C}^\times$ by $\phi(x) = e^{\frac{i2\pi \text{Tr}(x)}{p}}$, so that ϕ is a nontrivial character from the additive group of \mathbb{F}_q to the multiplicative group \mathbb{C}^\times . We note that $\ker \phi = \ker \text{Tr}$. For $a \in \mathbb{F}_q$, we define $\phi_a \in \text{Irr}(\mathbb{F}_q)$ by $\phi_a(t) = \phi(at)$, and note that $\text{Irr}(\mathbb{F}_q) = \{\phi_a \mid a \in \mathbb{F}_q\}$.

It is clear that $\text{Tr}(a_1 s_1 + \cdots + a_r s_r) = 0$ for all $s_1, \dots, s_r \in \mathbb{F}_q$ holds if and only if $a_1 = \cdots = a_r = 0$. Moreover, since the Frobenius automorphism $t \mapsto t^p$ is an automorphism of \mathbb{F}_q , we have that the equality $\text{Tr}(at^p) = 0$ holds for all $t \in \mathbb{F}_q$ if and only if $a = 0$.

The next lemma is important in our analysis of nonabelian cores; a version of this lemma giving $\ker \phi$ for an arbitrary choice of character $\phi : \mathbb{F}_q \rightarrow \mathbb{C}^\times$, which is less explicit, was proved in [LM1, Proposition 1.3].

Lemma 2.2. *For a fixed $a \in \mathbb{F}_q^\times$, let $\mathbb{T}_a = \{t^p - a^{p-1}t \mid t \in \mathbb{F}_q\}$. Then*

$$a^{-p} \mathbb{T}_a = \ker \text{Tr}.$$

Proof. We have that

$$a^{-p} \mathbb{T}_a = \{a^{-p}(t^p - a^{p-1}t) \mid t \in \mathbb{F}_q\} = \{(ta^{-1})^p - ta^{-1} \mid t \in \mathbb{F}_q\} = \{u^p - u \mid u \in \mathbb{F}_q\}.$$

Now, we also have that

$$\text{Tr}(t^p - t) = \text{Tr}(t^p) - \text{Tr}(t) = \text{Tr}(t) - \text{Tr}(t) = 0.$$

Therefore,

$$\{t^p - t \mid t \in \mathbb{F}_q\} \subseteq \{x \in \mathbb{F}_q \mid \text{Tr}(x) = 0\} = \ker \text{Tr},$$

and all those sets have same cardinality q/p , therefore $\ker(\text{Tr}) = \{t^p - t \mid t \in \mathbb{F}_q\} = a^{-p} \mathbb{T}_a$. \square

2.2. Background on reductive groups. We introduce now the main notation for finite reductive groups that we require. We cite [DM, Section 3], as a reference for the theory of algebraic groups over finite fields, and for the terminology used here.

Let \mathbf{G} be a connected reductive algebraic group defined and split over \mathbb{F}_p . We assume that p is not a very bad prime for \mathbf{G} ; recall that this means that $p > 2$ if \mathbf{G} is of type B_r , C_r or F_4 , and $p > 3$ if \mathbf{G} is of type G_2 .

Fix \mathbf{B} a Borel subgroup of \mathbf{G} defined over \mathbb{F}_p , and let \mathbf{T} be a maximal torus of \mathbf{G} contained in \mathbf{B} and defined over \mathbb{F}_p . We write \mathbf{U} for the unipotent radical of \mathbf{B} , which is defined over \mathbb{F}_p . For a subgroup \mathbf{H} of \mathbf{G} defined over \mathbb{F}_p , we write $H = \mathbf{H}(q)$ for the group of \mathbb{F}_q -rational points of \mathbf{H} . So $G = \mathbf{G}(q)$ is a finite Chevalley group and $U = \mathbf{U}(q)$ is a Sylow p -subgroup of G .

For G of type X_r , we sometimes write U_{X_r} instead of just U , so that we can discuss different groups at the same time.

We denote by Φ the root system of \mathbf{G} with respect to \mathbf{T} , and by Φ^+ the set of positive roots in Φ determined by \mathbf{B} . Let $N = |\Phi^+|$. Recall that the standard (strict) partial order on Φ is defined by $\alpha < \beta$ if $\beta - \alpha$ is a sum of positive roots. We fix an enumeration of $\Phi^+ = \{\alpha_1, \dots, \alpha_N\}$ such that $i < j$ whenever $\alpha_i < \alpha_j$.

For $\alpha \in \Phi^+$, we write X_α for the corresponding root subgroup of U and we choose an isomorphism $x_\alpha : \mathbb{F}_q \rightarrow X_\alpha$. We abbreviate and write X_i for X_{α_i} and x_i for x_{α_i} . Each element of U can be written uniquely as $u = x_1(s_1)x_2(s_2)\cdots x_N(s_N)$, where $s_i \in \mathbb{F}_q$ for all $i = 1, \dots, N$. In particular, the X_i generate U , and $|U| = q^N$.

We now recall some standard facts about the commutator relations in U , we refer the reader to [Ca, Chapters 4 and 5] for more details. Given $\alpha, \beta \in \Phi^+$, we have

$$[x_\alpha(r), x_\beta(s)] = \prod_{\substack{i,j>0: \\ i\alpha+j\beta \in \Phi^+}} x_{i\alpha+j\beta}(c_{ij}^{\alpha,\beta} r^i s^j)$$

for certain coefficients $c_{ij}^{\alpha,\beta} \in \mathbb{F}_p$. The parameterizations of the root subgroups can be chosen so that the coefficients $c_{ij}^{\alpha,\beta}$ are always $\pm 1, \pm 2, \pm 3$, where ± 2 occurs only for G of type B_r, C_r, F_4 and G_2 , and ± 3 only occurs for G of type G_2 . Moreover, as p is not very bad for G , we have that

$$[X_\alpha, X_\beta] = \prod_{\substack{i,j>0: \\ i\alpha+j\beta \in \Phi^+}} X_{i\alpha+j\beta}$$

for $\alpha, \beta \in \Phi^+$.

2.3. Quattern groups. In our algorithm for determining the irreducible characters of U , we require certain subquotients of U , which we refer to as quattern groups. The term pattern subgroup that we use below goes back to Isaacs, [Is2, Section 3], and quattern groups were also used in [HLM2]. We give the required terminology and notation here. Most of the assertions made here are well-known, proofs can be found for example in [HLM2, Sections 3 and 4].

A subset \mathcal{P} of Φ^+ is said to be *closed* (or a *pattern*) if for $\alpha, \beta \in \mathcal{P}$, we have that $\alpha + \beta \in \mathcal{P}$ whenever $\alpha + \beta \in \Phi^+$. For a closed subset \mathcal{P} of Φ^+ , we say that $\mathcal{K} \subseteq \mathcal{P}$ is *normal in \mathcal{P}* , and write $\mathcal{K} \trianglelefteq \mathcal{P}$, if for all $\delta \in \mathcal{K}$ and $\alpha \in \mathcal{P}$, we have $\delta + \alpha \in \mathcal{K}$ whenever $\delta + \alpha \in \Phi^+$. A subset \mathcal{S} of Φ^+ is called a *quattern* if $\mathcal{S} = \mathcal{P} \setminus \mathcal{K}$, where \mathcal{P} is closed and \mathcal{K} is normal in \mathcal{P} .

Let \mathcal{P} be a closed subset Φ^+ , let \mathcal{K} be normal in \mathcal{P} , and let $\mathcal{S} = \mathcal{P} \setminus \mathcal{K}$. We define

$$X_{\mathcal{P}} = \prod_{\alpha \in \mathcal{P}} X_\alpha.$$

It is a straightforward exercise using the commutator relations to show that $X_{\mathcal{P}}$ is a subgroup of U . We refer to a subgroup of U of the form $X_{\mathcal{P}}$ as a *pattern group*. Further, it is a consequence of the commutator relations that $X_{\mathcal{K}}$ is a normal subgroup of $X_{\mathcal{P}}$, and we define

$$X_{\mathcal{S}} = X_{\mathcal{P} \setminus \mathcal{K}} = X_{\mathcal{P}} / X_{\mathcal{K}}.$$

It follows from the construction of $X_{\mathcal{S}}$ that the natural map $\prod_{\alpha \in \mathcal{S}} X_\alpha \rightarrow X_{\mathcal{S}}$ is a bijection.

A subquotient of U of the form $X_{\mathcal{S}}$ is called a *quattern group*. We can easily check that $X_{\mathcal{S}}$ is independent up to (canonical) isomorphism of the possible choice of \mathcal{P} and \mathcal{K} such that $\mathcal{S} = \mathcal{P} \setminus \mathcal{K}$, so there is no ambiguity in the notation $X_{\mathcal{S}}$. We write $\mathcal{S} = \mathcal{P} \setminus \mathcal{K}$ for a

quaternion, where we are implicitly assuming that \mathcal{P} and \mathcal{K} are such a choice. Given $\alpha \in \mathcal{S}$, by a mild abuse of notation we identify X_α with its image in $X_{\mathcal{S}}$ for the remainder of this paper.

Let $\mathcal{S} \subseteq \Phi^+$ be a quaternion and let $X_{\mathcal{S}}$ be the corresponding quaternion group. We define

$$\mathcal{Z}(\mathcal{S}) = \{\gamma \in \mathcal{S} \mid \gamma + \alpha \notin \mathcal{S} \text{ for all } \alpha \in \mathcal{S}\}$$

and

$$\mathcal{D}(\mathcal{S}) = \{\gamma \in \mathcal{Z}(\mathcal{S}) \mid \alpha + \beta \neq \gamma \text{ for all } \alpha, \beta \in \mathcal{S}\}.$$

Using the commutator relations and the assumption that p is not very bad for G , it can be shown that

$$Z(X_{\mathcal{S}}) = X_{\mathcal{Z}(\mathcal{S})}.$$

Then it can be seen that

$$X_{\mathcal{S}} \cong X_{\mathcal{S} \setminus \mathcal{D}(\mathcal{S})} \times X_{\mathcal{D}(\mathcal{S})}.$$

Let \mathcal{S} be a quaternion and let $\mathcal{Z} \subseteq \mathcal{Z}(\mathcal{S})$. We define

$$\text{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} = \{\chi \in \text{Irr}(X_{\mathcal{S}}) \mid X_\alpha \not\subseteq \ker(\chi) \text{ for all } \alpha \in \mathcal{Z}\}.$$

These sets of irreducible characters are key to the algorithm presented in the next section.

Next we recall that a subset Σ of Φ^+ is called an *antichain* if for all $\alpha, \beta \in \Sigma$, we have $\alpha \not\leq \beta$ and $\beta \not\leq \alpha$, i.e. α and β are incomparable in the partial order on Φ^+ .

Given an antichain Σ in Φ^+ , the set $\mathcal{K}_\Sigma = \{\beta \in \Phi^+ \mid \beta \not\leq \gamma \text{ for all } \gamma \in \Sigma\}$ is a normal subset of Φ^+ . Conversely, given a normal subset \mathcal{K} of Φ^+ the set $\Sigma_{\mathcal{K}}$ of maximal elements of $\Phi^+ \setminus \mathcal{K}$ is clearly an antichain in Φ^+ . This sets up a bijective correspondence between antichains in Φ^+ and normal subsets in Φ^+ . The assertions made above are standard properties of posets, see for example [CP, Section 4].

For an antichain Σ in Φ^+ , we define the quaternion $\mathcal{S}_\Sigma = \Phi^+ \setminus \mathcal{K}_\Sigma$. Then it is an easy consequence of the definitions that $\mathcal{Z}(\mathcal{S}_\Sigma) = \Sigma$.

Now let $\chi \in \text{Irr}(U)$. We define $\mathcal{R}(\chi) = \{\alpha \in \Phi^+ \mid X_\alpha \subseteq \ker \chi\}$. Using the commutator relations it is easy to see that $\mathcal{R}(\chi)$ is a normal subset of Φ^+ , and thus $\Sigma_{\mathcal{R}(\chi)}$ is an antichain in Φ^+ . For an antichain $\Sigma \in \Phi^+$, we define $\text{Irr}(U)_\Sigma = \{\chi \in \text{Irr}(U) \mid \Sigma_{\mathcal{R}(\chi)} = \Sigma\}$. Then clearly we have the partition

$$\text{Irr}(U) = \bigsqcup_{\Sigma} \text{Irr}(U)_\Sigma,$$

where the union is taken over all antichains Σ in Φ^+ . Moreover, we have that any character in $\text{Irr}(U)_\Sigma$ is the inflation of an irreducible character in $\text{Irr}(X_{\mathcal{S}_\Sigma})_\Sigma$.

We frequently want to inflate and induce characters from one quaternion group to another, so we fix some notation for this. Let $\mathcal{S}' = \mathcal{P}' \setminus \mathcal{K}'$ and $\mathcal{S} = \mathcal{P} \setminus \mathcal{K}$ be quaternions, and let ψ be a character of $X_{\mathcal{S}'}$. If $\mathcal{P}' = \mathcal{P}$ and $\mathcal{K}' \supseteq \mathcal{K}$, then we let $\mathcal{L} = \mathcal{K}' \setminus \mathcal{K}$ and we write $\text{Inf}_{\mathcal{L}} \psi = \text{Inf}_{X_{\mathcal{S}'}}^{X_{\mathcal{S}}} \psi$ for the inflation of ψ from $X_{\mathcal{S}'}$ to $X_{\mathcal{S}}$; in case $\mathcal{L} = \{\alpha\}$ has one element, we write $\text{Inf}_\alpha \psi = \text{Inf}_{\mathcal{L}} \psi$. If $\mathcal{K}' = \mathcal{K}$ and $\mathcal{P}' \subseteq \mathcal{P}$, then we let $\mathcal{T} = \mathcal{P} \setminus \mathcal{P}'$ and we write $\text{Ind}^{\mathcal{T}} \psi$ for $\text{Ind}_{X_{\mathcal{S}'}}^{X_{\mathcal{S}}} \psi$; in case $\mathcal{T} = \{\alpha\}$ has one element, we write $\text{Ind}^\alpha \psi$ for $\text{Ind}^{\mathcal{T}} \psi$.

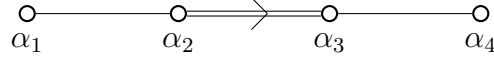


FIGURE 1. The Dynkin diagram of a root system of type F_4 .

2.4. Notation for F_4 . We fix some specific notation in the case \mathbf{G} is of type F_4 that we use later. In this case the Dynkin diagram of Φ is given in Figure 1, where $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is the set of simple roots determined by Φ^+ .

There are 24 positive roots in Φ , listed in Table 1; they are enumerated as in CHEVIE [CHEVIE]. We use the notation $\begin{smallmatrix} 2 & 3 & 4 & 2 \end{smallmatrix}$ for the root $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4$, and similar notation for the other positive roots. The roots are enumerated so that their height is nondecreasing; we recall that the *height* of $\sum_{i=1}^4 a_i \alpha_i$ is by definition $\sum_{i=1}^4 a_i$. We choose parameterizations of the root subgroups in U so that the commutator relations are given as in Table 2; all $[x_i(s), x_j(r)]$ not listed in this table are equal to 1.

Height	Roots			
1	α_1	α_2	α_3	α_4
2	$\alpha_5 = 1 \ 1 \ 0 \ 0$	$\alpha_6 = 0 \ 1 \ 1 \ 0$	$\alpha_7 = 0 \ 0 \ 1 \ 1$	
3	$\alpha_8 = 1 \ 1 \ 1 \ 0$	$\alpha_9 = 0 \ 1 \ 2 \ 0$	$\alpha_{10} = 0 \ 1 \ 1 \ 1$	
4	$\alpha_{11} = 1 \ 1 \ 2 \ 0$	$\alpha_{12} = 1 \ 1 \ 1 \ 1$	$\alpha_{13} = 0 \ 1 \ 2 \ 1$	
5	$\alpha_{14} = 1 \ 2 \ 2 \ 0$	$\alpha_{15} = 1 \ 1 \ 2 \ 1$	$\alpha_{16} = 0 \ 1 \ 2 \ 2$	
6	$\alpha_{17} = 1 \ 2 \ 2 \ 1$	$\alpha_{18} = 1 \ 1 \ 2 \ 2$		
7	$\alpha_{19} = 1 \ 2 \ 3 \ 1$	$\alpha_{20} = 1 \ 2 \ 2 \ 2$		
8	$\alpha_{21} = 1 \ 2 \ 3 \ 2$			
9	$\alpha_{22} = 1 \ 2 \ 4 \ 2$			
10	$\alpha_{23} = 1 \ 3 \ 4 \ 2$			
11	$\alpha_{24} = 2 \ 3 \ 4 \ 2$			

TABLE 1. Positive roots in a root system of type F_4 .

3. ALGORITHM TO PARAMETERIZE THE IRREDUCIBLE CHARACTERS OF U

3.1. Lemmas required for the algorithm. Before describing our algorithm to determine the irreducible characters of U , we present a couple of lemmas which are the basis of the reductions performed in the algorithm.

Our first lemma gives a specific version of Lemma 2.1.

Lemma 3.1. *Let $\mathcal{S} = \mathcal{P} \setminus \mathcal{K}$ be a quattern, let $\mathcal{Z} \subseteq \mathcal{Z}(\mathcal{S})$ and let $\gamma \in \mathcal{Z}$. Suppose that there exist $\delta, \beta \in \mathcal{S} \setminus \{\gamma\}$, with $\delta + \beta = \gamma$, such that for all $\alpha, \alpha' \in \mathcal{S}$ we have $\alpha + \alpha' \neq \beta$, and that for all $\alpha \in \mathcal{S} \setminus \{\beta\}$ we have $\delta + \alpha \notin \mathcal{S}$. Let $\mathcal{P}' = \mathcal{P} \setminus \{\beta\}$ and $\mathcal{K}' = \mathcal{K} \cup \{\delta\}$. Then we have*

$[x_1(s), x_2(r)] = x_5(rs)$	$[x_1(s), x_6(r)] = x_8(rs)x_{14}(-r^2s)$
$[x_1(s), x_9(r)] = x_{11}(rs)$	$[x_1(s), x_{10}(r)] = x_{12}(rs)x_{20}(r^2s)$
$[x_1(s), x_{13}(r)] = x_{15}(rs)x_{22}(r^2s)$	$[x_1(s), x_{16}(r)] = x_{18}(rs)$
$[x_1(s), x_{23}(r)] = x_{24}(rs)$	$[x_2(s), x_3(r)] = x_6(rs)x_9(-r^2s)$
$[x_2(s), x_7(r)] = x_{10}(rs)x_{16}(r^2s)$	$[x_2(s), x_{11}(r)] = x_{14}(rs)$
$[x_2(s), x_{15}(r)] = x_{17}(rs)x_{24}(r^2s)$	$[x_2(s), x_{18}(r)] = x_{20}(rs)$
$[x_2(s), x_{22}(r)] = x_{23}(rs)$	$[x_3(s), x_4(r)] = x_7(rs)$
$[x_3(s), x_5(r)] = x_8(-rs)x_{11}(rs^2)$	$[x_3(s), x_6(r)] = x_9(2rs)$
$[x_3(s), x_8(r)] = x_{11}(2rs)$	$[x_3(s), x_{10}(r)] = x_{13}(rs)$
$[x_3(s), x_{12}(r)] = x_{15}(rs)$	$[x_3(s), x_{17}(r)] = x_{19}(rs)$
$[x_3(s), x_{20}(r)] = x_{21}(rs)x_{22}(-rs^2)$	$[x_3(s), x_{21}(r)] = x_{22}(2rs)$
$[x_4(s), x_6(r)] = x_{10}(-rs)$	$[x_4(s), x_8(r)] = x_{12}(-rs)$
$[x_4(s), x_9(r)] = x_{13}(-rs)x_{16}(rs^2)$	$[x_4(s), x_{11}(r)] = x_{15}(-rs)x_{18}(rs^2)$
$[x_4(s), x_{13}(r)] = x_{16}(2rs)$	$[x_4(s), x_{14}(r)] = x_{17}(-rs)x_{20}(rs^2)$
$[x_4(s), x_{15}(r)] = x_{18}(2rs)$	$[x_4(s), x_{17}(r)] = x_{20}(2rs)$
$[x_4(s), x_{19}(r)] = x_{21}(rs)$	$[x_5(s), x_7(r)] = x_{12}(rs)x_{18}(r^2s)$
$[x_5(s), x_9(r)] = x_{14}(-rs)$	$[x_5(s), x_{13}(r)] = x_{17}(-rs)x_{23}(-r^2s)$
$[x_5(s), x_{16}(r)] = x_{20}(-rs)$	$[x_5(s), x_{22}(r)] = x_{24}(rs)$
$[x_6(s), x_7(r)] = x_{13}(-rs)$	$[x_6(s), x_8(r)] = x_{14}(2rs)$
$[x_6(s), x_{12}(r)] = x_{17}(rs)$	$[x_6(s), x_{15}(r)] = x_{19}(-rs)$
$[x_6(s), x_{18}(r)] = x_{21}(-rs)x_{23}(rs^2)$	$[x_6(s), x_{21}(r)] = x_{23}(2rs)$
$[x_7(s), x_8(r)] = x_{15}(rs)$	$[x_7(s), x_{10}(r)] = x_{16}(-2rs)$
$[x_7(s), x_{12}(r)] = x_{18}(-2rs)$	$[x_7(s), x_{14}(r)] = x_{19}(-rs)x_{22}(rs^2)$
$[x_7(s), x_{17}(r)] = x_{21}(rs)$	$[x_7(s), x_{19}(r)] = x_{22}(2rs)$
$[x_8(s), x_{10}(r)] = x_{17}(-rs)$	$[x_8(s), x_{13}(r)] = x_{19}(rs)$
$[x_8(s), x_{16}(r)] = x_{21}(rs)x_{24}(-rs^2)$	$[x_8(s), x_{21}(r)] = x_{24}(2rs)$
$[x_9(s), x_{12}(r)] = x_{19}(rs)x_{24}(-r^2s)$	$[x_9(s), x_{18}(r)] = x_{22}(-rs)$
$[x_9(s), x_{20}(r)] = x_{23}(-rs)$	$[x_{10}(s), x_{11}(r)] = x_{19}(rs)x_{23}(-rs^2)$
$[x_{10}(s), x_{12}(r)] = x_{20}(-2rs)$	$[x_{10}(s), x_{15}(r)] = x_{21}(-rs)$
$[x_{10}(s), x_{19}(r)] = x_{23}(2rs)$	$[x_{11}(s), x_{16}(r)] = x_{22}(rs)$
$[x_{11}(s), x_{20}(r)] = x_{24}(-rs)$	$[x_{12}(s), x_{13}(r)] = x_{21}(rs)$
$[x_{12}(s), x_{19}(r)] = x_{24}(2rs)$	$[x_{13}(s), x_{15}(r)] = x_{22}(-2rs)$
$[x_{13}(s), x_{17}(r)] = x_{23}(-2rs)$	$[x_{14}(s), x_{16}(r)] = x_{23}(rs)$
$[x_{14}(s), x_{18}(r)] = x_{24}(rs)$	$[x_{15}(s), x_{17}(r)] = x_{24}(-2rs)$

TABLE 2. Commutator relations for U for G of type F_4 .

that $\mathcal{S}' = \mathcal{P}' \setminus \mathcal{K}'$ is a quattern with $X_{\mathcal{S}'} \cong X_{\mathcal{P}'} / X_{\mathcal{K}'}$, and we have a bijection

$$\begin{aligned} \text{Irr}(X_{\mathcal{S}'})_{\mathcal{Z}} &\rightarrow \text{Irr}(X_{\mathcal{S}})_{\mathcal{Z}} \\ \chi &\mapsto \text{Ind}^{\beta} \text{Inf}_{\delta} \chi \end{aligned}$$

by inflating over X_{δ} and inducing to $X_{\mathcal{S}}$ over X_{β} .

Proof. Let $\alpha, \alpha' \in \mathcal{P}'$. If $\alpha \in \mathcal{K}$ or $\alpha' \in \mathcal{K}$, then it cannot be that $\alpha + \alpha' = \beta$, since in that case we would get $\beta \in \mathcal{K}$, a contradiction with $\beta \in \mathcal{S}$. If $\alpha, \alpha' \in \mathcal{S}'$, by assumption the equality $\alpha + \alpha' = \beta$ cannot hold as well. Since $\mathcal{P}' = \mathcal{S}' \cup \mathcal{K}$, this proves that \mathcal{P}' is closed.

Let now $\alpha \in \mathcal{P}'$, and $\alpha' \in \mathcal{K}'$. If $\alpha' \in \mathcal{K}$, then $\alpha + \alpha' \in \mathcal{K}'$ whenever $\alpha + \alpha' \in \Phi^+$, since $\mathcal{K} \subseteq \mathcal{P}$. Otherwise, $\alpha' = \delta$, and by assumption $\alpha + \delta \notin \mathcal{S}$ since $\alpha \neq \beta$, therefore if $\alpha + \delta \in \Phi^+$ then $\alpha + \delta \in \mathcal{K}'$. Therefore $\mathcal{K}' \subseteq \mathcal{P}'$, and $\mathcal{S}' = \mathcal{P}' \setminus \mathcal{K}'$ is a quattern.

It is immediate that conditions (i)–(iv) of Lemma 2.1 hold with $G = X_{\mathcal{S}}$, $Z = X_{\gamma}$, $H = X_{\mathcal{S} \setminus \{\beta\}}$, $X = X_{\beta}$ and $Y = X_{\delta}$.

Let $\lambda \in \text{Irr}(Z)$ and $\tilde{\lambda} = \text{Inf}_{\delta} \lambda$. Then for $s_1, s_2 \in \mathbb{F}_q$, we have

$$x_{\beta(s_1)} \tilde{\lambda} = x_{\beta(s_2)} \tilde{\lambda} \text{ if and only if } \lambda([x_{\beta(s_1)}, x_{\delta}(t)]) = \lambda([x_{\beta(s_2)}, x_{\delta}(t)]) \text{ for all } t \in \mathbb{F}_q.$$

Therefore, the commutator formulas in §2.2 imply that condition (v) must be satisfied, and of course $|X| = |Y| = q$, so the lemma follows. \square

Our second lemma is an immediate consequence of the definitions. We state it for ease of reference later, and omit any proof.

Lemma 3.2. *Let \mathcal{S} be a quattern and let $\alpha \in \mathcal{Z}(\mathcal{S})$. Then there is a bijection $\text{Irr}(X_{\mathcal{S}}) \rightarrow \text{Irr}(X_{\mathcal{S}})_{\{\alpha\}} \sqcup \text{Irr}(X_{\mathcal{S} \setminus \{\alpha\}})$.*

3.2. An example of the algorithm. Before we give a description of our algorithm, we illustrate it in an example. We consider a case for G of type F_4 and use the notation given in §2.4.

We want to compute $\text{Irr}(U)_{\Sigma}$, where $\Sigma = \{\alpha_{12}\}$. We let $\mathcal{S} = \mathcal{S}_{\Sigma} = \Phi^+ \setminus \mathcal{K}_{\Sigma}$, so $\mathcal{S} = \{\alpha_1, \dots, \alpha_8\} \cup \{\alpha_{10}, \alpha_{12}\}$. Also we let $\mathcal{Z} = \Sigma = \{\alpha_{12}\}$. So we want to compute $\text{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}$.

Let

$$(\beta_1, \delta_1) = (\alpha_1, \alpha_{10}), \quad (\beta_2, \delta_2) = (\alpha_4, \alpha_8), \quad (\beta_3, \delta_3) = (\alpha_5, \alpha_7).$$

An application of Lemma 3.1, for $(\beta, \delta) = (\beta_1, \delta_1)$ gives a bijection $\text{Irr}(X_{\mathcal{S}^1})_{\mathcal{Z}} \rightarrow \text{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}$, where $\mathcal{S}^1 = \mathcal{S} \setminus \{\beta_1, \delta_1\}$. Two further applications give bijections $\text{Irr}(X_{\mathcal{S}^2})_{\mathcal{Z}} \rightarrow \text{Irr}(X_{\mathcal{S}^1})_{\mathcal{Z}}$ and $\text{Irr}(X_{\mathcal{S}^3})_{\mathcal{Z}} \rightarrow \text{Irr}(X_{\mathcal{S}^2})_{\mathcal{Z}}$, where $\mathcal{S}^2 = \mathcal{S}^1 \setminus \{\beta_2, \delta_2\}$ and $\mathcal{S}^3 = \mathcal{S}^2 \setminus \{\beta_3, \delta_3\}$. We record the sets $\mathcal{A} = \{\beta_1, \beta_2, \beta_3\}$ and $\mathcal{L} = \{\delta_1, \delta_2, \delta_3\}$ to remind us which reductions were performed. We also define $\mathcal{K} = \mathcal{K}_{\Sigma} \cup \mathcal{L}$. These three reductions are all instances of TYPE R reductions (the capitalized R means “reduction lemma”) in Algorithm 3.3 in the next subsection.

Now we can see that $\alpha_{12} \in \mathcal{D}(\mathcal{S}^3)$, so that $X_{\mathcal{S}^3} \cong X_{\mathcal{S}^3 \setminus \{\alpha_{12}\}} \times X_{12}$. In particular, this means there is no possibility to apply Lemma 3.1, with $\gamma \in \mathcal{Z} = \{\alpha_{12}\}$.

We find that $\mathcal{Z}(\mathcal{S}^3) \setminus \mathcal{D}(\mathcal{S}^3) = \{\alpha_6\}$. We can apply Lemma 3.2 to obtain a bijection

$$\text{Irr}(X_{\mathcal{S}^3})_{\mathcal{Z}} \rightarrow \text{Irr}(X_{\mathcal{S}^3})_{\mathcal{Z} \cup \{\alpha_6\}} \sqcup \text{Irr}(X_{\mathcal{S}^3 \setminus \{\alpha_6\}}).$$

We now split the two cases and consider them in turn. We note that this is an example of a TYPE S reduction (the capitalized S means “split”) as defined in our algorithm in the next subsection.

First we consider

$$\text{Irr}(X_{\mathcal{S}^3})_{\mathcal{Z}^3},$$

where $\mathcal{S}^3 = \{\alpha_2, \alpha_3, \alpha_6, \alpha_{12}\}$ and $\mathcal{Z}^3 = \{\alpha_6, \alpha_{12}\}$. We can apply Lemma 3.1 with $\delta = \alpha_3$, $\beta = \alpha_2$, and $\gamma = \alpha_6$. We then get a bijection $\text{Irr}(X_{\mathcal{S}^4})_{\mathcal{Z}^3} \rightarrow \text{Irr}(X_{\mathcal{S}^3})_{\mathcal{Z}^3}$, where $\mathcal{S}^4 = \mathcal{S}^3 \setminus \{\alpha_2, \alpha_3\} = \{\alpha_6, \alpha_{12}\}$. This is another reduction of TYPE R as defined in the next subsection. We record this reduction by adjoining α_2 to \mathcal{A} to obtain $\mathcal{A}' = \{\alpha_1, \alpha_4, \alpha_5, \alpha_2\}$ and adjoining α_3 to \mathcal{L} to obtain $\mathcal{L}' = \{\alpha_{10}, \alpha_8, \alpha_7, \alpha_3\}$. Moreover, we put $\mathcal{K}' = \mathcal{K}_{\Sigma} \cup \mathcal{L}'$.

We note that $X_{\mathcal{S}^4} = X_6 \times X_{12}$, so we can parameterize $\text{Irr}(X_{\mathcal{S}^4})_{\mathcal{Z}^3}$ as $\{\lambda^{a_6, a_{12}} \mid a_6, a_{12} \in \mathbb{F}_q^{\times}\}$, where $\lambda^{a_6, a_{12}}(x_6(t)) = \phi(a_6 t)$ and $\lambda^{a_6, a_{12}}(x_{12}(t)) = \phi(a_{12} t)$. Through the bijections given by

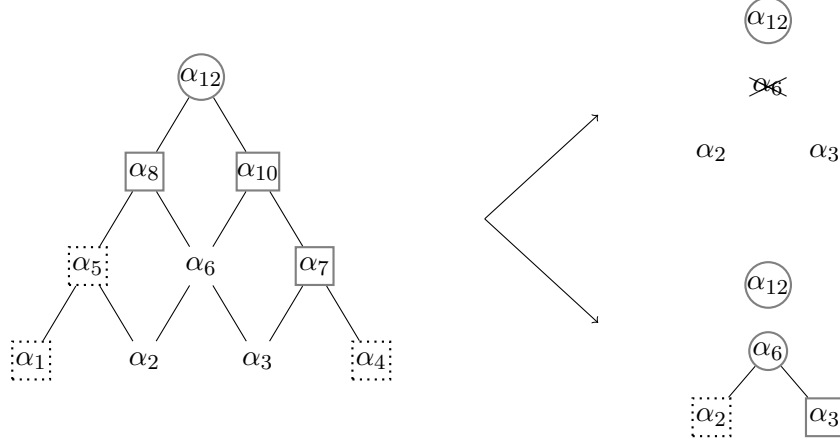


FIGURE 2. A pictorial representation of the calculation of the characters in $\text{Irr}(U)_{\{\alpha_{12}\}}$ for G of type F_4 .

Lemma 3.1, we obtain characters of U forming part of $\text{Irr}(U)_\Sigma$ by a process of successive inflation and induction of the characters $\lambda^{a_6, a_{12}}$. These characters are

$$\chi^{a_6, a_{12}} = \text{Inf}_{\mathcal{K}_\Sigma} \text{Ind}^{\alpha_1} \text{Inf}_{\alpha_{10}} \text{Ind}^{\alpha_4} \text{Inf}_{\alpha_8} \text{Ind}^{\alpha_5} \text{Inf}_{\alpha_7} \text{Ind}^{\alpha_2} \text{Inf}_{\alpha_3} \lambda^{a_6, a_{12}}.$$

However, it turns out that these characters can be obtained by a single inflation and then induction, thanks to Theorem 3.8, and we have

$$\chi^{a_6, a_{12}} = \text{Ind}^{\mathcal{A}'} \text{Inf}_{\mathcal{K}'} \lambda^{a_6, a_{12}}.$$

The characters $\chi^{a_6, a_{12}}$ have degree q^4 .

Next we move on to consider the characters in $\text{Irr}(X_{S^5})_{\mathcal{Z}}$ where $\mathcal{S}^5 = \mathcal{S}^3 \setminus \{\alpha_6\} = \{\alpha_2, \alpha_3, \alpha_{12}\}$, and $\mathcal{Z} = \{\alpha_{12}\}$. We record that we have put α_6 in the kernel by adjoining it to \mathcal{K} to obtain $\mathcal{K}'' = \mathcal{K} \cup \{\alpha_6\}$. We see that X_{S^5} is abelian, so that $\text{Irr}(X_{S^5}) = \{\lambda_{b_2, b_3}^{a_{12}} \mid a_{12} \in \mathbb{F}_q^\times, b_2, b_3 \in \mathbb{F}_q\}$, where $\lambda_{b_2, b_3}^{a_{12}}(x_2(t)) = \phi(b_2 t)$ and $\lambda_{b_2, b_3}^{a_{12}}(x_3(t)) = \phi(b_3 t)$, and $\lambda_{b_2, b_3}^{a_{12}}(x_{12}(t)) = \phi(a_{12} t)$.

Now through the bijections previously obtained in Lemma 3.1, we obtain characters $\chi_{b_2, b_3}^{a_{12}}$ of U forming part of $\text{Irr}(U)_\Sigma$ from the characters $\lambda_{b_2, b_3}^{a_{12}}$ by a process of successive inflation and induction. We have

$$\chi_{b_2, b_3}^{a_{12}} = \text{Inf}_{\mathcal{K}_\Sigma} \text{Ind}^{\alpha_1} \text{Inf}_{\alpha_{10}} \text{Ind}^{\alpha_4} \text{Inf}_{\alpha_8} \text{Ind}^{\alpha_5} \text{Inf}_{\alpha_7} \text{Inf}_{\alpha_6} \lambda_{b_2, b_3}^{a_{12}},$$

and note that by using Theorem 3.8, we can write these characters as

$$\chi_{b_2, b_3}^{a_{12}} = \text{Ind}^{\mathcal{A}} \text{Inf}_{\mathcal{K}''} \lambda^{a_6, a_{12}}.$$

These characters have degree q^3 .

Putting this together, we have that

$$\text{Irr}(U)_{\{\alpha_{12}\}} = \{\chi^{a_6, a_{12}} \mid a_6, a_{12} \in \mathbb{F}_q^\times\} \sqcup \{\chi_{b_2, b_3}^{a_{12}} \mid b_2, b_3 \in \mathbb{F}_q, a_{12} \in \mathbb{F}_q^\times\}.$$

Therefore, $\text{Irr}(U)_{\{\alpha_{12}\}}$ consists of:

- $(q-1)^2$ characters of degree q^4 ; and
- $q^2(q-1)$ characters of degree q^3 .

In Figure 2, we illustrate how we have calculated these characters. The roots in a circle are in \mathcal{Z} ; the roots in a straight box are in \mathcal{L} and the roots in a dotted box are in \mathcal{A} .

3.3. The algorithm. Our algorithm is used to calculate $\text{Irr}(U)_\Sigma$ for each antichain Σ in Φ^+ . We explain the algorithm below, which is written in a sort of pseudocode; the comments in *italics* aim to make it easier to understand.

Algorithm 3.3.

INPUT:

- $\Phi^+ = \{\alpha_1, \dots, \alpha_N\}$, the set of positive roots of a root system with a fixed enumeration such that $i \leq j$ whenever $\alpha_i \leq \alpha_j$.
- Σ , an antichain in Φ^+ .

VARIABLES:

- $\mathcal{S} \subseteq \Phi^+$ is a quattern.
- \mathcal{Z} is a subset of $\mathcal{Z}(\mathcal{S})$.
- $\mathcal{A} \subseteq \Phi^+$ keeps a record of the roots β used in a TYPE R reduction.
- $\mathcal{L} \subseteq \Phi^+$ keeps a record of the roots δ used in a TYPE R reduction.
- $\mathcal{K} \subseteq \Phi^+$ keeps a record of the roots indexing root subgroups in the quotient of the associated quattern group.
- \mathfrak{S} is a stack of tuples of the form $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ as above to be considered later in the algorithm.
- $\mathfrak{D} = (\mathfrak{D}_1, \mathfrak{D}_2)$ is the output.

INITIALIZATION:

- $\mathcal{K} := \mathcal{K}_\Sigma$.
- $\mathcal{S} := \Phi^+ \setminus \mathcal{K}_\Sigma$.
- $\mathcal{Z} := \Sigma$.
- $\mathcal{A} := \emptyset$.
- $\mathcal{L} := \emptyset$.
- $\mathfrak{S} := \emptyset$.
- $\mathfrak{D} := (\emptyset, \emptyset)$.

During the algorithm we consider $\text{Irr}(X_\mathcal{S})_\mathcal{Z}$, going into four subroutines called “ABELIAN CORE”, “TYPE R”, “TYPE S” and “NONABELIAN CORE”.

ABELIAN CORE.

if $\mathcal{S} = \mathcal{Z}(\mathcal{S})$ **then**

Adjoin $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ to \mathfrak{D}_1 .

In this case $X_\mathcal{S}$ is abelian and we can parameterize the characters in $\text{Irr}(X_\mathcal{S})_\mathcal{Z}$.

if $\mathfrak{S} = \emptyset$ **then**

Finish and **output** \mathfrak{D} .

In this case we have no more characters to consider, so we are done.

else

Remove the tuple at the top of the stack \mathfrak{S} and replace $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ with it, and go to ABELIAN CORE.

end if

else

Go to TYPE R.

end if

TYPE R.

Look for pairs $(\beta, \delta) = (\alpha_i, \alpha_j)$ that satisfy the conditions of Lemma 3.1 for some $\gamma \in \mathcal{Z}$.

if such a pair (α_i, α_j) exists **then**

Choose the pair with j maximal, and update the variables as follows.

- $\mathcal{S} := \mathcal{S} \setminus \{\alpha_i, \alpha_j\}$.
- $\mathcal{A} := \mathcal{A} \cup \{\alpha_i\}$.
- $\mathcal{L} := \mathcal{L} \cup \{\alpha_j\}$.
- $\mathcal{K} := \mathcal{K} \cup \{\alpha_j\}$.

We are replacing \mathcal{S} with \mathcal{S}' as in Lemma 3.1, and recording this in \mathcal{A} , \mathcal{L} and \mathcal{K} .

Go to ABELIAN CORE.

else

Go to TYPE S.

end if

TYPE S.

if $\mathcal{Z}(\mathcal{S}) \setminus (\mathcal{Z} \cup \mathcal{D}(\mathcal{S})) \neq \emptyset$ **then**

Let i be maximal such that $\alpha_i \in \mathcal{Z}(\mathcal{S}) \setminus (\mathcal{Z} \cup \mathcal{D}(\mathcal{S}))$, and update as follows.

- $\mathfrak{S} := \mathfrak{S} \cup \{(\mathcal{S} \setminus \{\alpha_i\}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K} \cup \{\alpha_i\})\}$.
- $\mathcal{Z} := \mathcal{Z} \cup \{\alpha_i\}$.

Here we are using Lemma 3.2. We first add $(\mathcal{S} \setminus \{\alpha_i\}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K} \cup \{\alpha_i\})$ to the stack to be considered later, recording that X_{α_i} is in the kernel of these characters by adding α_i to \mathcal{K} . Then we replace $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ with $(\mathcal{S}, \mathcal{Z} \cup \{\alpha_i\}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ for the current run.

Go to ABELIAN CORE.

else

Go to NONABELIAN CORE

end if

NONABELIAN CORE.

Adjoin $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ to \mathfrak{D}_2 .

We are no longer able to apply reductions of TYPE R or of TYPE S, and $X_{\mathcal{S}}$ is not abelian, so the algorithm gives up, and this case is output as a nonabelian core as discussed further later.

if $\mathfrak{S} = \emptyset$ **then**

Output \mathfrak{D} and finish.

In this case we have no more characters to consider, so we are done.

else

Remove the tuple at the top of the stack \mathfrak{S} and replace $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ with it, and go to ABELIAN CORE.

end if

The letter R in the TYPE R reduction means “reduction lemma”, while the letter S in TYPE S means “split”. The letters \mathcal{A} and \mathcal{L} mean “arm” and “leg” respectively; this

terminology is used in [HLM2], and it is motivated by the fact that each pair (β, δ) gives rise to a so-called “hook” subgroup.

We move on to discuss how we interpret the output. We begin by defining what we mean by a *core*, which is an element of the output of our algorithm.

Definition 3.4. Let us suppose that Algorithm 3.3 has run with input (Φ^+, Σ) and given output \mathfrak{O} .

- An element $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ of \mathfrak{O}_1 is called an *abelian core* for $\text{Irr}(U)_\Sigma$.
- An element $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ of \mathfrak{O}_2 is called a *nonabelian core* for $\text{Irr}(U)_\Sigma$.

We discuss how we can determine the characters in $\text{Irr}(U)_\Sigma$ corresponding to a core $\mathfrak{C} = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ in $\mathfrak{O}_1 \cup \mathfrak{O}_2$. In particular, when $\mathfrak{C} \in \mathfrak{O}_1$ is an abelian core, we give a complete description of the irreducible characters, however for nonabelian cores there is more work required. We require some notation for what occurs in the algorithm.

We obtain \mathfrak{C} through a sequence of reductions of TYPE R and of TYPE S applied in Algorithm 3.3; though here we only need to consider the TYPE S reduction in this sequence if a root γ is added to \mathcal{K} (rather than to \mathcal{Z}). So we consider the sequence of reductions where in each one either:

- a pair of roots β and δ is taken from \mathcal{S} in a TYPE R reduction, and β is added to \mathcal{A} and δ is added to \mathcal{L} and \mathcal{K} ; or
- a root γ is taken from \mathcal{S} and added to \mathcal{K} .

We let $\ell = \ell_{\mathfrak{C}}$ be the number of these reductions, and define the sequence $T(\mathfrak{C}) = (t_1, \dots, t_\ell)$, where $t_i = \text{R}$ if the i th reduction is a TYPE R reduction and $t_i = \text{S}$ if the i th reduction is a TYPE S reduction. We let $I(\text{R}, \mathfrak{C})$ be the set of i such that $t_i = \text{R}$ and $I(\text{S}, \mathfrak{C})$ be the set of i such that $t_i = \text{S}$. For $i \in I(\text{R}, \mathfrak{C})$ we write (β_i, δ_i) for the pair of roots used in the TYPE R reduction, and for $i \in I(\text{S}, \mathfrak{C})$, we write γ_i for the root added to \mathcal{K} in the TYPE S reduction. Thus we have $\mathcal{A} = \{\beta_i \mid i \in I(\text{R}, \mathfrak{C})\}$, $\mathcal{L} = \{\delta_i \mid i \in I(\text{R}, \mathfrak{C})\}$ and $\mathcal{K} \setminus \mathcal{K}_\Sigma = \mathcal{L} \cup \{\gamma_i \mid i \in I(\text{S}, \mathfrak{C})\}$.

We also define the subsets $\mathcal{P}^0, \mathcal{P}^1, \dots, \mathcal{P}^\ell$ and $\mathcal{K}^0, \mathcal{K}^1, \dots, \mathcal{K}^\ell$ of Φ^+ recursively by

$$\begin{aligned} \mathcal{P}^0 &= \Phi^+ \text{ and } \mathcal{K}^0 = \mathcal{K}_\Sigma; \\ \mathcal{P}^i &= \begin{cases} \mathcal{P}^{i-1} \setminus \{\beta_i\} & \text{if } t_i = \text{R} \\ \mathcal{P}^{i-1} & \text{if } t_i = \text{S} \end{cases} \\ \mathcal{K}^i &= \begin{cases} \mathcal{K}^{i-1} \cup \{\delta_i\} & \text{if } t_i = \text{R} \\ \mathcal{K}^{i-1} \cup \{\gamma_i\} & \text{if } t_i = \text{S} \end{cases} \end{aligned}$$

We have the following lemma about these sets.

Lemma 3.5. *For each $i, j = 1, \dots, \ell$ with $i \leq j$, we have that \mathcal{P}^j is a closed set, and \mathcal{K}^i is normal in \mathcal{P}^j . In particular, $\mathcal{S}^{i,j} = \mathcal{P}^j \setminus \mathcal{K}^i$ are quatters.*

Proof. Of course, $\mathcal{P}^0 = \Phi^+$ is closed. Let us assume that \mathcal{P}^{i-1} is closed. Without loss of generality, let $\mathcal{P}^i = \mathcal{P}^{i-1} \setminus \{\beta_i\}$. For $\alpha, \alpha' \in \mathcal{P}^i$, it cannot be that $\alpha + \alpha' = \beta_i$ by construction of \mathcal{P}^i . Also, by inductive assumption, we have that $\alpha + \alpha' \in \mathcal{P}^{i-1}$ if $\alpha + \alpha'$ is a positive root. This implies $\alpha + \alpha' \in \mathcal{P}^i$ or $\alpha + \alpha' \notin \Phi^+$, that is, \mathcal{P}^i is closed.

To prove that \mathcal{K}^i is normal in \mathcal{P}^j for $i \leq j$, it is enough to prove that \mathcal{K}^i is normal in \mathcal{P}^i , since $\mathcal{K}^i \subseteq \mathcal{P}^j \subseteq \mathcal{P}^i$. Let $\alpha \in \mathcal{P}^i$ and $\eta \in \mathcal{K}^i$. Recall that $\eta \in \mathcal{K}_\Sigma$ or η is of the form γ_k or δ_k as above for some $k \leq i$. If $\eta \in \mathcal{K}_\Sigma$, then since $\mathcal{K}_\Sigma \trianglelefteq \Phi^+$ we have that $\alpha + \eta \in \mathcal{K}_\Sigma$ whenever

$\alpha + \eta \in \Phi^+$. If $\eta = \gamma_k$ for some $k \leq i$, then η is a central root in $\mathcal{S}^{k-1, k-1} \supseteq \mathcal{S}^{i, i}$, therefore since $\alpha \in \mathcal{P}^i$ we have that $\alpha + \eta \in \mathcal{K}^{k-1} \subseteq \mathcal{K}^i$ or $\alpha + \eta \notin \Phi^+$. If $\eta = \delta_k$, then we notice that $\beta_k \notin \mathcal{P}^k$, thus $\beta_k \notin \mathcal{P}^i$, therefore if $\alpha \in \mathcal{P}^i$ then $\alpha + \eta \in \mathcal{K}^{k-1}$ or $\alpha + \eta \notin \Phi^+$. This implies that \mathcal{K}^i is normal in \mathcal{P}^i . \square

Let $\psi \in \text{Irr}(X_S)$. We define characters $\overline{\psi}_i \in \text{Irr}(X_{\mathcal{P}^i \setminus \mathcal{K}^i})$ for $i = \ell, \ell-1, \dots, 1, 0$ recursively by the following sequence of inflations and inductions.

$$\begin{aligned} \overline{\psi}_\ell &= \psi \\ \overline{\psi}_{i-1} &= \begin{cases} \text{Ind}^{\beta_i} \text{Inf}_{\delta_i} \overline{\psi}_i & \text{if } t_i = R \\ \text{Inf}_{\gamma_i} \overline{\psi}_i & \text{if } t_i = S \end{cases} \end{aligned}$$

Finally, we let $\overline{\psi} = \text{Inf}_{\mathcal{K}_\Sigma} \overline{\psi}_0 \in \text{Irr}(U)$.

Suppose that $\mathfrak{C} = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K}) \in \mathfrak{D}_1$ is an abelian core. We let $\mathcal{Z} = \{\alpha_{i_1}, \dots, \alpha_{i_m}\}$ and $\mathcal{S} \setminus \mathcal{Z} = \{\alpha_{j_1}, \dots, \alpha_{j_n}\}$. Then we have

$$\text{Irr}(X_S)_\mathcal{Z} = \{\lambda_{\underline{b}}^{\underline{a}} \mid \underline{a} = (a_{i_1}, \dots, a_{i_m}) \in (\mathbb{F}_q^\times)^m, \underline{b} = (b_{j_1}, \dots, b_{j_n}) \in \mathbb{F}_q^n\},$$

where $\lambda_{\underline{b}}^{\underline{a}}$ is defined by

$$\lambda_{\underline{b}}^{\underline{a}}(x_{\alpha_{i_k}}(t)) = \phi(a_{i_k} t) \quad \text{and} \quad \lambda_{\underline{b}}^{\underline{a}}(x_{\alpha_{j_h}}(t)) = \phi(b_{j_h} t)$$

for every $k = 1, \dots, m$ and $h = 1, \dots, n$. We define $\chi_{\underline{b}}^{\underline{a}} = \overline{\lambda_{\underline{b}}^{\underline{a}}}$ and

$$\text{Irr}(U)_\mathfrak{C} = \{\chi_{\underline{b}}^{\underline{a}} \mid \underline{a} = (a_{i_1}, \dots, a_{i_m}) \in (\mathbb{F}_q^\times)^m, \underline{b} = (b_{j_1}, \dots, b_{j_n}) \in \mathbb{F}_q^n\}.$$

Through the bijections given by Lemmas 3.1 and 3.2, this is precisely the set of characters in $\text{Irr}(U)_\Sigma$ corresponding to \mathfrak{C} .

We move on to consider a nonabelian core $\mathfrak{C} = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K}) \in \mathfrak{D}_2$. In this case X_S is not abelian, so we do not immediately have a parametrization of $\text{Irr}(X_S)_\mathcal{Z}$, and it is necessary for us to determine a parametrization by hand. We suppose this has been done and we have

$$\text{Irr}(X_S)_\mathcal{Z} = \{\psi_{\underline{c}} \mid \underline{c} \in J_{\mathfrak{C}}\},$$

where $J_{\mathfrak{C}}$ is some indexing set. We define $\chi_{\underline{c}} = \overline{\psi_{\underline{c}}}$ and

$$\text{Irr}(U)_\mathfrak{C} = \{\chi_{\underline{c}} \mid \underline{c} \in J_{\mathfrak{C}}\}.$$

The aim of the next section is to develop a method towards a determination of the set $J_{\mathfrak{C}}$ when \mathfrak{C} is a nonabelian core.

From the comments given within Algorithm 3.3 and the discussion above, we deduce the following theorem regarding the validity of our algorithm.

Theorem 3.6. *Suppose that Algorithm 3.3 has run with input (Φ^+, Σ) and given output $\mathfrak{D} = (\mathfrak{D}_1, \mathfrak{D}_2)$. Then we have*

$$\text{Irr}(U)_\Sigma = \bigsqcup_{\mathfrak{C} \in \mathfrak{D}_1} \text{Irr}(U)_\mathfrak{C} \sqcup \bigsqcup_{\mathfrak{C} \in \mathfrak{D}_2} \text{Irr}(U)_\mathfrak{C}.$$

We note that the definitions of $\chi_{\underline{b}}^{\underline{a}}$ and $\chi_{\underline{c}}$ given above involve a potentially very long sequence of inflations and inductions. In fact it turns out that we can obtain them by a single inflation followed by a single induction, which is stated in Theorem 3.8 below.

To prove this theorem, we require the following lemma. In the statement of the lemma, we use the notation $\mathcal{A}_i = \{\beta_j \mid j \geq i\}$, $\mathcal{L}_i = \{\delta_j \mid j \geq i\}$ and $\mathcal{K}_i = \{\gamma_j \mid j \geq i\}$.

Lemma 3.7. *Let $\psi \in \text{Irr}(X_S)$, and for $i = 0, 1, \dots, \ell$ define ψ_i as above. Then we have*

$$\psi_i = \text{Ind}^{\mathcal{A}_i} \text{Inf}_{\mathcal{L}_i \cup \mathcal{K}_i} \psi.$$

Proof. We prove this by reverse induction on i , the case $i = \ell$ being trivial.

The inductive step boils down to showing that

$$\text{Inf}_{\delta_i} \text{Ind}^{\mathcal{A}_{i+1}} = \text{Ind}^{\mathcal{A}_{i+1}} \text{Inf}_{\delta_i}$$

if $t_i = R$ and showing that

$$\text{Inf}_{\gamma_i} \text{Ind}^{\mathcal{A}_{i+1}} = \text{Ind}^{\mathcal{A}_{i+1}} \text{Inf}_{\gamma_i}$$

if $t_i = S$. Thanks to Lemma 3.5, we are able to apply (2.1) to deduce both of these equalities. \square

Theorem 3.8.

- (a) *Let $\mathfrak{C} \in (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K}) \in \mathfrak{D}_1$ be an abelian core, and let $\chi_{\underline{b}}^a \in \text{Irr}(U)_{\mathfrak{C}}$ be defined as above. Then*

$$\chi_{\underline{b}}^a = \text{Ind}^{\mathcal{A}} \text{Inf}_{\mathcal{K}} \lambda_{\underline{b}}^a.$$

In particular, $\chi_{\underline{b}}^a$ is induced from a linear character of $X_{S \cup \mathcal{K}}$.

- (b) *Let $\mathfrak{C} \in (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K}) \in \mathfrak{D}_2$ be a nonabelian core, and let $\chi_{\underline{c}} \in \text{Irr}(U)_{\mathfrak{C}}$ be defined as above. Then*

$$\chi_{\underline{c}} = \text{Ind}^{\mathcal{A}} \text{Inf}_{\mathcal{K}} \psi_{\underline{c}}.$$

Proof. We only prove (a) as the proof of (b) is entirely similar.

By Lemma 3.7, we have that $(\chi_{\underline{b}}^a)_0 = \text{Ind}^{\mathcal{A}} \text{Inf}_{(\mathcal{K} \setminus \mathcal{K}_{\Sigma})} \lambda_{\underline{b}}^a$. Thus

$$\chi_{\underline{b}}^a = \text{Inf}_{\mathcal{K}_{\Sigma}} \text{Ind}^{\mathcal{A}} \text{Inf}_{(\mathcal{K} \setminus \mathcal{K}_{\Sigma})} \lambda_{\underline{b}}^a.$$

Now we can apply (2.1) to see that $\text{Inf}_{\mathcal{K}_{\Sigma}} \text{Ind}^{\mathcal{A}} = \text{Ind}^{\mathcal{A}} \text{Inf}_{\mathcal{K}_{\Sigma}}$ from which we can deduce the theorem. \square

Remark 3.9. The choice of total order on $\Phi^+ = \{\alpha_1, \dots, \alpha_N\}$ has a significant effect on how the algorithm runs, as this is used to determine which reductions to make when there may be a choice. The resulting parametrization of $\text{Irr}(U)_{\Sigma}$ consequently depends on this choice of enumeration.

Remark 3.10. We make a slight abuse in the notation $\chi_{\underline{b}}^a$. In fact, each a_i and b_j is supposed to record not just a value in \mathbb{F}_q^{\times} and \mathbb{F}_q respectively, but also i and j , so that $\chi_{\underline{b}}^a$ should strictly read $\chi_{((j_1, b_{j_1}), \dots, (j_n, b_{j_n}))}^{((i_1, a_{i_1}), \dots, (i_m, a_{i_m}))}$, for corresponding choices of i_1, \dots, i_m and j_1, \dots, j_n indexing positive roots.

3.4. Results of algorithm. We have implemented Algorithm 3.3 in the algebra system GAP3 [GAP3], using the CHEVIE package [CHEVIE]. The algorithm requires us to just work with Φ^+ and the GAP commands for root systems allow us to do this. We use the enumeration of Φ^+ as given in GAP.

We have run the GAP program for G of rank less than or equal to 7. For ranks less than or equal to 4 we are able to deduce a complete parametrization, as the number of nonabelian cores is low. More specifically, for G of rank 3 or less, or G of type C_4 , there are no nonabelian cores, whilst for the types B_4 and D_4 there is one nonabelian core each, and in type F_4 we find six nonabelian cores. The nonabelian core for type D_4 has already

been encountered in [HLM1], and the core for type B_4 has the same representation theory for $p \neq 2$ as the one in type D_4 , so these cores have been analysed. The nonabelian cores for type F_4 are analysed in §4.3, and the corresponding irreducible characters are determined. The resulting parameterizations of irreducible characters of U_{B_4} , U_{C_4} and U_{F_4} are tabulated in the appendix. The parametrization for U_{D_4} is contained in [HLM1]. Also we note that the parametrization of irreducible characters for U_{B_3} can be read off from that for U_{B_4} , as U_{B_3} is a quotient of U_{B_4} . Similarly, the parametrization of irreducible characters of U_{C_3} can be read off from that for U_{C_4} .

From the parameterizations we can determine the number of irreducible characters of U of a given fixed degree. In particular, we observe that if G is of rank at most 4 and p is good, then all irreducible characters of U are of degree q^d for some $d \in \mathbb{Z}_{\geq 0}$. Moreover, the numbers of irreducible characters of U of degree q^d , is given by a polynomial in q and these polynomials are the same as the ones given in [GMR2, Table 3], where they were only known to be valid for $p \geq h$ (the Coxeter number of G). Further we also obtain expressions as polynomials in q for the number of characters of a given degree for type F_4 , and $p = 3$; these are given in Table 3. The case of type D_4 and $p = 2$ is covered in [HLM1].

For G of rank greater than 4, the number of nonabelian cores grows, so it is necessary to develop an approach to deal with these in a systematic way. This should be based on our analysis of nonabelian cores in the next section and is a topic for future research. In Table 4, we present the output from the algorithm including the number of nonabelian cores.

4. NONABELIAN CORES

In this section we explain the methods we employ to analyse nonabelian cores. It is helpful for us first to deal with certain 3-dimensional groups that arise in our analysis. Then we outline our general method to deal with nonabelian cores, before explaining how this is applied to the nonabelian cores in types B_4 and F_4 .

4.1. Some 3-dimensional groups. Let $f : \mathbb{F}_q \times \mathbb{F}_q \rightarrow \mathbb{F}_q$ be an \mathbb{F}_p -bilinear map, which we assume to be surjective. We define the group $V = V_f$ to be generated by subgroups $X_1 = \{x_1(t) \mid t \in \mathbb{F}_q\} \cong \mathbb{F}_q$, $X_2 = \{x_2(t) \mid t \in \mathbb{F}_q\} \cong \mathbb{F}_q$ and $Z = \{z(t) \mid t \in \mathbb{F}_q\} \cong \mathbb{F}_q$, subject to $Z \subseteq Z(V)$ and $[x_1(s), x_2(t)] = z(f(s, t))$. In particular, throughout this subsection, X_1 will not denote the root subgroup X_{α_1} , similarly for X_2 . It is straightforward to see that V is a nilpotent group and that $V = X_1 X_2 Z$. Moreover, our assumption that f is surjective implies that the derived subgroup of V is Z . We note that V is not necessarily a special group as its center can be strictly larger than its derived subgroup.

We quickly explain how to construct the irreducible characters of V .

First we note that the linear characters are given by the characters of $V/Z \cong X_1 \times X_2$. For $b_1, b_2 \in \mathbb{F}_q$, we define $\chi_{b_1, b_2} \in \text{Irr}(V)$ by $\chi_{b_1, b_2}(x_1(s_1)x_2(s_2)z(t)) = \phi(b_1 s_1 + b_2 s_2)$, so that the linear characters of V are $\{\chi_{b_1, b_2} \mid b_1, b_2 \in \mathbb{F}_q\}$.

For each $a \in \mathbb{F}_q^\times$, we define the linear character $\lambda^a \in \text{Irr}(Z)$ by $\lambda^a(z(t)) = \phi(at)$. We analyse the characters in $\text{Irr}(V \mid \lambda^a)$ using Lemma 2.1. We define $X'_1 = \{x_1(s) \in X_1 \mid \text{Tr}(af(s, t)) = 0 \text{ for all } t \in \mathbb{F}_q\}$ and define X'_2 similarly. Note that X'_1 and X'_2 may depend on a . Since the map $\mathbb{F}_q \times \mathbb{F}_q \rightarrow \mathbb{F}_p$ given by $(s, t) \mapsto \text{Tr}(f(s, t))$ is \mathbb{F}_p -bilinear, we deduce that X'_1 and X'_2 are \mathbb{F}_p -subspaces of $X_1 \cong \mathbb{F}_q$ and $X_2 \cong \mathbb{F}_q$ respectively, and we have $|X'_1| = |X'_2|$. Thus we can choose complements \tilde{X}_1 and \tilde{X}_2 of X'_1 and X'_2 in X_1 and X_2 respectively. Now

d	$k(U_{F_4}(q), d)$
1	$v^4 + 4v^3 + 6v^2 + 4v + 1$
q	$v^5 + 6v^4 + 13v^3 + 12v^2 + 4v$
q^2	$v^6 + 7v^5 + 20v^4 + 28v^3 + 18v^2 + 4v$
q^3	$4v^5 + 20v^4 + 33v^3 + 21v^2 + 4v$
$q^4/3$	0, if $p \geq 5$ $9v^4/2$, if $p = 3$
q^4	$v^8 + 8v^7 + 28v^6 + 58v^5 + 79v^4 + 66v^3 + 24v^2 + 2v$, if $p \geq 5$ $v^8 + 8v^7 + 28v^6 + 59v^5 + 161v^4/2 + 67v^3 + 24v^2 + 2v$, if $p = 3$
q^5	$v^7 + 7v^6 + 22v^5 + 39v^4 + 37v^3 + 15v^2 + 2v$, if $p \geq 5$ $v^7 + 7v^6 + 23v^5 + 41v^4 + 37v^3 + 15v^2 + 2v$, if $p = 3$
q^6	$2v^6 + 14v^5 + 36v^4 + 40v^3 + 17v^2 + 2v$, if $p \geq 5$ $2v^6 + 14v^5 + 36v^4 + 39v^3 + 17v^2 + 2v$, if $p = 3$
q^7	$2v^6 + 13v^5 + 32v^4 + 34v^3 + 13v^2 + 2v$
q^8	$4v^5 + 15v^4 + 19v^3 + 8v^2$
q^9	$v^5 + 7v^4 + 11v^3 + 5v^2$
q^{10}	$v^4 + 3v^3 + v^2$
$k(U_{F_4}(q)) = \begin{cases} v^8 + 9v^7 + 40v^6 + 124v^5 + 256v^4 + 288v^3 + 140v^2 + 24v + 1, & \text{if } p \geq 5 \\ v^8 + 9v^7 + 40v^6 + 126v^5 + 264v^4 + 288v^3 + 140v^2 + 24v + 1, & \text{if } p = 3 \end{cases}$	

TABLE 3. Numbers of irreducible characters of U_{F_4} of fixed degree, for $v = q - 1$ and $p \neq 2$.

by Lemma 2.1 (with $X = \tilde{X}_1$ and $Y = \tilde{Y}_1$), we see that $\psi \mapsto \text{Ind}_{X'_1 X_2 Z}^V \text{Inf}_{X'_1 X_2 Z / (\tilde{X}_2 \ker \lambda^a)}^{X'_1 X_2 Z} \psi$ gives a bijection from $\text{Irr}(X'_1 X_2 Z / (\tilde{X}_2 \ker \lambda^a) \mid \lambda^a)$ to $\text{Irr}(V \mid \lambda^a)$. Finally, we observe that $X'_1 X_2 Z / (\tilde{X}_2 \ker \lambda^a)$ is abelian, so $\text{Irr}(X'_1 X_2 Z / (\tilde{X}_2 \ker \lambda^a) \mid \lambda^a)$ is in bijection with the set $\text{Irr}(X'_1 \times X'_2)$, which is easily described.

Before considering some particular choices of f , we note that it is straightforward to show that $V_f \cong V_{af}$ for $a \in \mathbb{F}_q^\times$ by reparamaterizing Z appropriately.

For $f(s, t) = st$, we see that V_f is isomorphic to U_{A_2} . Clearly we get $X' = Y' = 1$ and then

$$\text{Irr}(V) = \{\chi_{b_1, b_2} \mid b_1, b_2 \in \mathbb{F}_q\} \cup \{\chi^a \mid a \in \mathbb{F}_q^\times\},$$

where $\chi^a = \text{Ind}_{X_2 Z}^V \text{Inf}_Z^{X_2 Z} \lambda^a$. Similarly for $f(s, t) = s^p t$ or $f(s, t) = (s^p - ds)t$ where $d \in \mathbb{F}_q^\times$ is not a $(p - 1)$ th power, we see that $X' = Y' = 1$, and $\text{Irr}(V)$ is given as above.

Type	Antichains	Abelian cores	Nonabelian cores	Running time
B ₄	70	80	1 (1.23%)	$T \ll 1$ sec
C ₄	70	90	0 (0%)	$T \ll 1$ sec
D ₄	50	52	1 (1.88%)	$T \ll 1$ sec
F ₄	105	177	6 (3.28%)	$T \sim 1$ sec
B ₅	252	358	10 (2.72%)	$T \sim 3$ sec
C ₅	252	417	1 (0.24%)	$T \sim 3$ sec
D ₅	182	214	7 (3.17%)	$T \sim 1$ sec
B ₆	924	1842	95 (4.90%)	$T \sim 30$ sec
C ₆	924	2254	22 (0.97%)	$T \sim 30$ sec
D ₆	672	991	55 (5.26%)	$T \sim 10$ sec
E ₆	833	1656	156 (8.61%)	$T \sim 30$ sec
B ₇	3432	11240	969 (7.94%)	$T \sim 7$ min
C ₇	3432	14216	294 (2.03%)	$T \sim 7$ min
D ₇	2508	5479	531 (8.84%)	$T \sim 2.5$ min
E ₇	4160	33594	7798 (18.84%)	$T \sim 10$ min

TABLE 4. Results of the algorithm applied in types B_i, C_i and D_i, $i = 4, 5, 6, 7$ and F₄, E_k, $k = 6, 7$.

The case of major interest to us here is $f(s, t) = (s^p - ds)t$ where $d \in \mathbb{F}_q^\times$ is a $(p-1)$ -th power, say $d = e^{p-1}$. Then we find that $X'_1 = \{x_1(s) \mid s^p - ds = 0\} = \{x_1(s) \mid s \in e\mathbb{F}_p\}$ and $X'_2 = \{x_2(t) \mid \text{Tr}(at\mathbb{T}_e) = 0\} = \{x_2(t) \mid t \in (e^{-p}/a)\mathbb{F}_p\}$ using Lemma 2.2. Now for $c_1, c_2 \in \mathbb{F}_p$ we define the characters $\lambda_{c_1, c_2}^a \in \text{Irr}(X'_1 X'_2 Z / (\tilde{X}_2 \ker \lambda^a))$ by

$$\lambda_{c_1, c_2}^a(x_1(es_1)x_2((e^{-p}/a)s_2)z(t)) = \phi(c_1s_1 + c_2s_2 + at).$$

for every $s_1, s_2 \in \mathbb{F}_p$ and $t \in \mathbb{F}_q$. Then we have

$$\text{Irr}(V) = \{\chi_{b_1, b_2} \mid b_1, b_2 \in \mathbb{F}_q\} \cup \{\chi_{c_1, c_2}^a \mid a \in \mathbb{F}_q^\times, c_1, c_2 \in \mathbb{F}_p\},$$

where $\chi_{c_1, c_2}^a = \text{Ind}_{X'_1 X'_2 Z}^V \text{Inf}_{X'_1 X'_2 Z / (\tilde{X}_2 \ker \lambda^a)}^{X'_1 X'_2 Z} \lambda_{c_1, c_2}^a$. In this case, we get q^2 linear characters and $p^2(q-1)$ characters of degree q/p .

4.2. A method for analysing nonabelian cores. Let $\mathfrak{C} = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ be a nonabelian core. The set \mathcal{S} is a quatern corresponding to the pattern $\Phi^+ \setminus \mathcal{A}$ and its normal subset \mathcal{K} . Further, we have $\mathcal{Z} = \mathcal{Z}(\mathcal{S}) \setminus \mathcal{D}(\mathcal{S})$ as \mathfrak{C} is a nonabelian core, and we let $\mathcal{Z} = \{\alpha_{i_1}, \dots, \alpha_{i_m}\}$. For each $\underline{a} = (a_{i_1}, \dots, a_{i_m}) \in (\mathbb{F}_q^\times)^m$, we define $\mu = \mu^{\underline{a}} : X_{\mathcal{Z}} \rightarrow \mathbb{F}_q$ by $\mu(x_{i_h}(t)) = a_{i_h}t$ for $h = 1, \dots, m$. Then $\lambda = \lambda^{\underline{a}} = \phi \circ \mu^{\underline{a}}$ is a linear character of $X_{\mathcal{Z}}$.

We give a method to analyse the characters in $\text{Irr}(X_{\mathcal{S}} \mid \lambda)$. We note that the nature of the resulting parametrization and construction of the characters may depend on the choice of \underline{a} , and we see instances of this dependence in §4.3. Further we remark that we do not assert that this method is guaranteed to work for every nonabelian core, though it does apply for all the cores that we consider in §4.3.

We set $V = X_{\mathcal{S}} / \ker \mu$ and $Z = X_{\mathcal{Z}} / \ker \mu$. Since $\ker \mu \subseteq \ker \lambda$, we have that λ factors through Z and we also write λ for this character of Z . Then we have a bijection between

$\text{Irr}(V \mid \lambda)$ and $\text{Irr}(X_{\mathcal{S}} \mid \lambda)$ by inflating over $\ker \mu$, and we work in $\text{Irr}(V \mid \lambda)$ rather than in $\text{Irr}(X_{\mathcal{S}} \mid \lambda)$. Given $\alpha \in \mathcal{S} \setminus \mathcal{Z}$ we identify X_{α} with its image in V .

We aim to find subsets \mathcal{I} and \mathcal{J} of $\mathcal{S} \setminus \mathcal{Z}$ such that the following hold.

- $|\mathcal{I}| = |\mathcal{J}|$;
- $H = X_{\mathcal{S} \setminus (\mathcal{I} \cup \mathcal{Z})} Z$ is a subgroup of V ;
- $Y = X_{\mathcal{J}} \leq Z(H)$; and
- YZ is a normal subgroup of V .

We note that this implies that

- $X = X_{\mathcal{I}}$ is a transversal of H in V .

We would like to apply Lemma 2.1 (the reduction lemma), and conditions (i)–(iv) do hold, but condition (v) may not be satisfied, so we aim to adapt the situation slightly.

We consider the inflation $\hat{\mu}$ of μ to YZ and let $\hat{\lambda} = \phi \circ \hat{\mu}$ be the inflation of λ to YZ . For $v \in V$, we consider the map $\psi_v : Y \rightarrow \mathbb{F}_q$ given by $\psi_v(y) = \hat{\mu}([v, y])$. Since $Y \leq Z(H)$ and $YZ \trianglelefteq V$, we deduce from the commutator relations that ψ_v is \mathbb{F}_q -linear. We let

$$Y' = \bigcap_{v \in V} \ker(\psi_v) = \{y \in Y \mid {}^v \hat{\mu}(y) = \hat{\mu}(y) \text{ for all } v \in V\}.$$

Then Y' is an \mathbb{F}_q -subspace of $Y \cong \mathbb{F}_q^{|\mathcal{J}|}$. Also, we define

$$\tilde{H} = \text{Stab}_V(\hat{\mu}) = \{v \in V \mid {}^v \hat{\mu} = \hat{\mu}\}.$$

Then \tilde{H} is a subgroup of V and $\tilde{H} = X'H$ for $X' = \{x \in X \mid {}^x \hat{\mu} = \hat{\mu}\}$.

To prove that X' and Y' have the same cardinality we assume, for the rest of this subsection, that

$$W = \{\psi_v \mid v \in V\} \text{ is an } \mathbb{F}_q\text{-subspace of the dual space } \text{Hom}(Y, \mathbb{F}_q).$$

This condition is easily checked to hold for all nonabelian cores that we examine when G is of rank 4 by looking at the form of (4.2) defined below in each of these cases.

Lemma 4.1. $|X'| = |Y'|$.

Proof. We have that the annihilator $\text{Ann}_Y(W)$ of W is Y' by definition. Hence we have $\dim Y = \dim Y' + \dim W$, that is, $|Y|/|Y'| = |W|$.

We denote the V -orbit of $\hat{\mu}$ in $\text{Hom}(YZ, \mathbb{F}_q)$ by $\hat{\mu}^V$. For $v, v' \in V$ we have that $\psi_v = \psi_{v'}$ if and only if $\hat{\mu}([v, y]) = \hat{\mu}([v', y])$ for all $y \in Y$ if and only if $\hat{\mu}(y^v) = \hat{\mu}(y^{v'})$ for all $y \in Y$. Then the map

$$\begin{aligned} \hat{\mu}^V &\longrightarrow W \\ \hat{\mu}^v &\mapsto \psi_v \end{aligned}$$

is well-defined and injective. It is clear that it is also surjective. Therefore, we have $|W| = |\hat{\mu}^V|$. Now by the orbit-stabilizer theorem we have that

$$|\hat{\mu}^V| = |V|/\text{Stab}_V(\hat{\mu}) = |V|/|\tilde{H}| = |X|/|X'|.$$

Combining the above equalities, we get

$$|Y|/|Y'| = |W| = |\hat{\mu}^V| = |X|/|X'|.$$

Since $|Y| = |X|$, the claim follows. □

Moreover, we have the following property about X' .

Lemma 4.2. *Let $x \in X$ be such that ${}^x\hat{\lambda} = \hat{\lambda}$. Then $x \in X'$.*

Proof. We show that for such x we have ${}^x\hat{\mu} = \hat{\mu}$. The hypothesis is equivalent to

$$\phi \circ {}^x\hat{\mu} = \phi \circ \hat{\mu}, \text{ that is, } \phi \circ ({}^x\hat{\mu} - \hat{\mu}) = 1. \quad (4.1)$$

For $y \in Y$ and $z \in Z$, we have

$${}^x\hat{\mu}(yz) = \hat{\mu}(y^x z^x) = {}^x\hat{\mu}(y) + \hat{\mu}(z) = \hat{\mu}([y, x]) + \mu(z) = -\psi_x(y) + \mu(z),$$

then using the assumptions that $Y \leq Z(H)$ and $YZ \trianglelefteq V$, we deduce from the commutator relations that ${}^x\hat{\mu}$ is \mathbb{F}_q -linear. Hence ${}^x\hat{\mu} - \hat{\mu}$ is also \mathbb{F}_q -linear. Therefore the image of ${}^x\hat{\mu} - \hat{\mu}$ is either 0 or \mathbb{F}_q . But if it were \mathbb{F}_q , then (4.1) would imply $\phi(c) = 1$ for every $c \in \mathbb{F}_q$, which is a contradiction. Thus we have ${}^x\hat{\mu} = \hat{\mu}$, so $x \in X'$. \square

We write $\mathcal{I} = \{\alpha_{i_1}, \dots, \alpha_{i_m}\}$ and $\mathcal{J} = \{\alpha_{j_1}, \dots, \alpha_{j_m}\}$, such that $i_1 \leq \dots \leq i_m$ and $j_1 \leq \dots \leq j_m$. In general, Y' and X' can be determined by the following equation,

$$\hat{\mu}([x_{\alpha_{j_1}}(s_{j_1}) \cdots x_{\alpha_{j_m}}(s_{j_m}), x_{\alpha_{i_1}}(t_{i_1}) \cdots x_{\alpha_{i_m}}(t_{i_m})]) = 0. \quad (4.2)$$

We note that as the map ψ_x for $x \in X$ is \mathbb{F}_q -linear, the left hand side of (4.2) is linear in s_{j_1}, \dots, s_{j_m} . Therefore, the values of s_{j_1}, \dots, s_{j_m} such that (4.2) holds for every t_{i_1}, \dots, t_{i_m} form an \mathbb{F}_q -subspace of Y , which determines Y' .

Under an additional assumption on Y , we are able to apply Lemma 2.1 in the following lemma. We define \tilde{H} to be the preimage of \tilde{H} in X_S .

Lemma 4.3. *Suppose that there exists a subgroup \tilde{Y} of Y such that $Y = Y' \times \tilde{Y}$ and $[X, \tilde{Y}] \subseteq \tilde{Y}Z$. Then we have a bijection*

$$\begin{aligned} \text{Irr}(\tilde{H}/\tilde{Y} \mid \lambda) &\rightarrow \text{Irr}(V \mid \lambda) \\ \chi &\mapsto \text{Ind}_{\tilde{H}}^V \text{Inf}_{\tilde{H}/\tilde{Y}}^{\tilde{H}} \chi \end{aligned}$$

Consequently we have a bijection

$$\begin{aligned} \text{Irr}(\tilde{H}/\tilde{Y} \mid \lambda) &\rightarrow \text{Irr}(X_S \mid \lambda) \\ \chi &\mapsto \text{Ind}_{\tilde{H}}^{X_S} \text{Inf}_{\tilde{H}/\tilde{Y}}^{\tilde{H}} \chi \end{aligned}$$

Proof. We want to check that $\tilde{H}, \tilde{X}, \tilde{Y}$ and Z satisfy all the assumptions of Lemma 2.1 as subgroups of V with respect to $\lambda \in \text{Irr}(Z)$. Clearly we have that $Z \leq Z(V)$ and $\tilde{Y} \cap Z = 1$. By assumption, we have that X normalizes $\tilde{Y}Z$, and we have that H centralizes $\tilde{Y}Z$, so $\tilde{Y}Z \trianglelefteq V$. Since $\tilde{Y} \leq Y \leq Z(H)$, we have that \tilde{Y} is normalized by H . Moreover, if $x' \in X'$ and $y \in Y$, by definition of X' we have that

$$\hat{\mu}(y^{-1}y^{x'}) = \hat{\mu}(y^{-1}) + \hat{\mu}(y^{x'}) = 0,$$

and since $\ker \hat{\mu} = Y \ker \mu$ we have that X' normalizes Y . Along with the assumption that $[X, \tilde{Y}] \subseteq \tilde{Y}Z$, we deduce that X' normalizes \tilde{Y} . Hence $\tilde{Y} \trianglelefteq \tilde{H}$.

Now we are left to check condition (v) of the reduction lemma. We write $\tilde{\lambda} \in \text{Irr}(\tilde{Y}Z)$ for the inflation of λ to $\tilde{Y}Z$, and note that $\tilde{\lambda} = \hat{\lambda}|_{\tilde{Y}Z}$. Let \tilde{X} be a transversal of \tilde{H} in V . Assume

that $\tilde{x}_1 \tilde{\lambda} = \tilde{x}_2 \tilde{\lambda}$ for $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$. Let $y \in Y$ and $z \in Z$ and write $y = y' \tilde{y}$, where $y' \in Y'$ and $\tilde{y} \in \tilde{Y}$. We have

$$\begin{aligned} \tilde{x}_1 \hat{\lambda}(y' \tilde{y} z) &= \hat{\lambda}(y'^{\tilde{x}_1}) \tilde{\lambda}(\tilde{y}^{\tilde{x}_1}) \lambda(z) \\ &= \hat{\lambda}(y') (\tilde{x}_1 \tilde{\lambda})(\tilde{y}) \lambda(z) \\ &= \hat{\lambda}(y'^{\tilde{x}_2}) (\tilde{x}_2 \tilde{\lambda})(\tilde{y}) \lambda(z) \\ &= \hat{\lambda}(y'^{\tilde{x}_2}) \hat{\lambda}(\tilde{y}^{\tilde{x}_2}) \lambda(z) \\ &= \tilde{x}_2 \hat{\lambda}(y' \tilde{y} z). \end{aligned}$$

In the above sequence of equalities we use that $\hat{\lambda}(y'^{\tilde{x}_1}) = \hat{\lambda}(y') = \hat{\lambda}(y'^{\tilde{x}_2})$ by definition of Y' , that $\tilde{y}^{\tilde{x}_1}, \tilde{y}^{\tilde{x}_2} \in \tilde{Y}Z$ since $[X, \tilde{Y}] \subseteq \tilde{Y}Z$, and that $\tilde{x}_1 \tilde{\lambda} = \tilde{x}_2 \tilde{\lambda}$ by assumption. Hence we have $\tilde{x}_1 \tilde{x}_2^{-1} \hat{\lambda} = \hat{\lambda}$. By Lemma 4.2, this implies $\tilde{x}_1 \tilde{x}_2^{-1} \in X'$ and thus $\tilde{x}_1 = \tilde{x}_2$ as \tilde{X} is a transversal of \tilde{H} in V . By Lemma 4.1, we have that $|X'| = |Y'|$. Thus we can apply Lemma 2.1 to deduce the first bijection.

We can now apply Lemma 2.1 also to deduce the second bijection. \square

We note that if $[X, Y] \subseteq Z$, then we may take an arbitrary complement \tilde{Y} of Y' in Y , and the assumption $[X, \tilde{Y}] \subseteq \tilde{Y}Z$ is obviously satisfied.

Also we note that the parametrization of characters resulting from Lemma 4.3 does not actually depend on the choice of \tilde{Y} . This can be shown by observing that the restriction of $\text{Ind}_{\tilde{H}}^V \text{Inf}_{\tilde{H}/\tilde{Y}}^{\tilde{H}} \chi$ to $Y'Z$ is a multiple of χ viewed as a character of $Y'Z$.

If Y' is central in \tilde{H}/\tilde{Y} , then we can extend $\lambda \in \text{Irr}(Z)$ to Y' . This turns out to be useful when applying the reduction lemma again in \tilde{H}/\tilde{Y} in the analysis in §4.3.

Remark 4.4. Suppose that Lemma 4.3 applies and let $\psi \in \text{Irr}(\tilde{H}/\tilde{Y} \mid \lambda)$. Then we have that $\text{Ind}_{\tilde{H}}^{X_S} \text{Inf}_{\tilde{H}/\tilde{Y}}^{\tilde{H}} \psi \in \text{Irr}(X_S)$, and

$$\overline{\psi} = \text{Ind}_{X_{S \cup K}}^U \text{Inf}_{X_S}^{X_{S \cup K}} \text{Ind}_{\tilde{H}}^{X_S} \text{Inf}_{\tilde{H}/\tilde{Y}}^{\tilde{H}} \psi \in \text{Irr}(U)_{\mathfrak{C}}$$

by Theorem 3.8. Since $X_K \trianglelefteq U$, we have that $\tilde{H}X_K$ is a subgroup of $X_{S \cup K}$, and we have $X_K \trianglelefteq \tilde{H}X_K$. Then using similar arguments to those in the proof of Lemma 3.7, we can show that

$$\overline{\psi} = \text{Ind}_{\tilde{H}X_K}^U \text{Inf}_{\tilde{H}/\tilde{Y}}^{\tilde{H}X_K} \psi.$$

In §4.3, we apply this argument (sometimes iteratively) to show that each irreducible character considered there can be obtained as an induced character of a linear character.

Consider the case that $Y' = 1$ for all choices of λ , and that Y is normal in \tilde{H} . We have $\tilde{H}/Y = X_{S \setminus (\mathcal{I} \cup \mathcal{J})}$. Defining

$$\overline{\psi} = \text{Ind}^{\mathcal{A} \cup \mathcal{I}} \text{Inf}_{K \cup \mathcal{J}} \psi$$

for $\psi \in \text{Irr}(X_{S \setminus (\mathcal{I} \cup \mathcal{J})})_{\mathcal{Z}}$ sets up a bijection from $\text{Irr}(X_{S \setminus (\mathcal{I} \cup \mathcal{J})})_{\mathcal{Z}}$ to $\text{Irr}(U)_{\mathfrak{C}}$.

4.3. The nonabelian cores for types B_4 and F_4 . For G of type B_4 there is one nonabelian core and for G of type F_4 , there are six nonabelian cores. We analyse these case by case using the method given in §4.2, and we use the notation introduced there.

For the nonabelian core in U_{B_4} and for one of the nonabelian cores in U_{F_4} , we find \mathcal{I} and \mathcal{J} such that $Y' = 1$ for all $\underline{a} \in (\mathbb{F}_q^\times)^m$, and Y is normal in \tilde{H} with \tilde{H}/Y abelian. For such a

core \mathfrak{C} we let $\mathcal{S} \setminus (\mathcal{I} \cup \mathcal{J}) = \{\alpha_{h_1}, \dots, \alpha_{h_n}\}$, and for $\underline{b} = (b_{h_1}, \dots, b_{h_n}) \in \mathbb{F}_q^n$, we let $\lambda_{\underline{b}}^{\underline{a}} \in \text{Irr}(H)$ be the linear character defined by $\lambda_{\underline{b}}^{\underline{a}}(x_{\alpha_{h_j}}(t)) = \phi(b_{h_j}t)$ for $j = 1, \dots, n$, and $\lambda_{\underline{b}}^{\underline{a}}|_{X_{\mathcal{Z}}} = \lambda^{\underline{a}}$. Let

$$\chi_{\underline{b}}^{\underline{a}} = \text{Ind}^{A \cup \mathcal{I}} \text{Inf}_{K \cup \mathcal{J}} \lambda_{\underline{b}}^{\underline{a}}.$$

Then using Remark 4.4, we see that

$$\text{Irr}(U)_{\mathfrak{C}} = \{\chi_{\underline{b}}^{\underline{a}} \mid \underline{a} \in (\mathbb{F}_q^\times)^m, \underline{b} \in \mathbb{F}_q^n\}.$$

For these cores we include no further details, and just give \mathcal{I} and \mathcal{J} in the tables in the appendix.

Below we consider the remaining nonabelian cores in $U = U_{F_4}$. We denote these cores by \mathfrak{C}^1 , \mathfrak{C}^2 , \mathfrak{C}^3 , \mathfrak{C}^4 and \mathfrak{C}^5 . For each $\mathfrak{C}^i = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ we give \mathcal{S} , \mathcal{Z} , \mathcal{A} and \mathcal{L} ; we note that \mathcal{K} can then easily be determined. Then we analyse $\text{Irr}(X_{\mathcal{S}})_{\mathcal{Z}}$ before explaining how this parameterizes $\text{Irr}(U)_{\mathfrak{C}^i}$ and how these characters can be obtained by inducing linear characters using Lemma 4.3 and Remark 4.4.

We notice that if $\mathfrak{C} = (\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$ and $\mathfrak{C}' = (\mathcal{S}', \mathcal{Z}', \mathcal{A}', \mathcal{L}', \mathcal{K}')$ are cores of U_{F_4} , then $(|\mathcal{S}|, |\mathcal{Z}|) \neq (|\mathcal{S}'|, |\mathcal{Z}'|)$. In particular, $X_{\mathcal{S}}$ is not isomorphic to $X_{\mathcal{S}'}$.

The nonabelian core in \mathfrak{C}^1 . This core occurs for $\Sigma = \{\alpha_{11}, \alpha_{13}\}$, and we have

- $\mathcal{S} = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{13}\}$,
- $\mathcal{Z} = \{\alpha_5, \alpha_{10}, \alpha_{11}, \alpha_{13}\}$,
- $\mathcal{A} = \{\alpha_3\}$ and
- $\mathcal{L} = \{\alpha_8\}$.

Using the method of §4.2, we take

- $Y = X_2 X_6 X_9$,
- $X = X_1 X_4 X_7$ and then we have that
- $H = YZ$.

In this case (4.2) is

$$s_2(-a_5 t_1 + a_{10} t_7) + s_6(a_{10} t_4 - a_{13} t_7) + s_9(-a_{11} t_1 + a_{13} t_4) = 0.$$

For $a_{11} \neq a_5 a_{13}^2 / a_{10}^2$, we have $Y' = 1$ and Y is normal in \bar{H} . Then as explained in Remark 4.4 we get the family of characters

$$\text{Irr}(U)_{\mathfrak{C}^1}^1 = \{\chi^{a_5, a_{10}, a_{11}^*, a_{13}} \mid a_5, a_{10}, a_{11}^*, a_{13} \in \mathbb{F}_q^\times, a_{11}^* \neq a_5(a_{13}/a_{10})^2\} \subseteq \text{Irr}(U)_{\mathfrak{C}^1},$$

where

$$\chi^{a_5, a_{10}, a_{11}^*, a_{13}} = \text{Ind}^{A \cup \mathcal{I}} \text{Inf}_{K \cup \mathcal{J}} \lambda^{a_5, a_{10}, a_{11}^*, a_{13}}.$$

We have that $\text{Irr}(U)_{\mathfrak{C}^1}^1$ consists of $(q-1)^3(q-2)$ characters of degree q^4 .

For $a_{11} = a_5 a_{13}^2 / a_{10}^2$, we have $X' = X_{1,4,7} = \{x_{1,4,7}(t) \mid t \in \mathbb{F}_q\}$ and $Y' = X_{2,6,9} = \{x_{2,6,9}(s) \mid s \in \mathbb{F}_q\}$, where

$$x_{1,4,7}(t) = x_1(a_{10}^2 t) x_4(a_5 a_{13} t) x_7(a_5 a_{10} t) \quad \text{and} \quad x_{2,6,9}(s) = x_2(a_{13}^2 s) x_6(a_{10} a_{13} s) x_9(-a_{10}^2 s).$$

We can take any complement of Y' in Y , and we choose $\tilde{Y} = X_2 X_9$. Then we have $\tilde{H}/\tilde{Y} = X' Y' Z$, which is abelian. We denote by $\lambda^{a_5, a_{10}, a_{13}}$ the character $\lambda^{a_5, a_{10}, a_{11}, a_{13}}$ with $a_{11} = a_5 a_{13}^2 / a_{10}^2$. For $b_{1,4,7}, b_{2,6,9} \in \mathbb{F}_q$, we define $\lambda_{b_{1,4,7}, b_{2,6,9}}^{a_5, a_{10}, a_{13}} \in \text{Irr}(X' Y' Z)$ by extending $\lambda^{a_5, a_{10}, a_{13}}$,

and setting $\lambda_{b_{1,4,7}, b_{2,6,9}}^{a_5, a_{10}, a_{13}}(x_{1,4,7}(t)) = \phi(b_{1,4,7}t)$ and $\lambda_{b_{1,4,7}, b_{2,6,9}}^{a_5, a_{10}, a_{13}}(x_{2,6,9}(t)) = \phi(b_{2,6,9}t)$ for every $t \in \mathbb{F}_q$. Then as explained in Remark 4.4 we get the family of characters

$$\text{Irr}(U)_{\mathfrak{C}^1}^2 = \{\chi_{b_{1,4,7}, b_{2,6,9}}^{a_5, a_{10}, a_{13}} \mid a_5, a_{10}, a_{13} \in \mathbb{F}_q^\times, b_{1,4,7}, b_{2,6,9} \in \mathbb{F}_q\},$$

where

$$\chi_{b_{1,4,7}, b_{2,6,9}}^{a_5, a_{10}, a_{13}} = \text{Ind}_{\bar{H}X_K}^U \text{Inf}_{\bar{H}/\bar{Y}}^{\bar{H}X_K} \lambda_{b_{1,4,7}, b_{2,6,9}}^{a_5, a_{10}, a_{13}}.$$

We have that $\text{Irr}(U)_{\mathfrak{C}^1}^2$ consists of $q^2(q-1)^3$ characters of degree q^3 .

We have that $\text{Irr}(U)_{\mathfrak{C}^1} = \text{Irr}(U)_{\mathfrak{C}^1}^1 \cup \text{Irr}(U)_{\mathfrak{C}^1}^2$ and this gives all the irreducible characters corresponding to \mathfrak{C}^1 .

The nonabelian core \mathfrak{C}^2 . This core occurs for $\Sigma = \{\alpha_{12}, \alpha_{16}\}$, and we have

- $\mathcal{S} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{12}, \alpha_{16}\},$
- $\mathcal{Z} = \{\alpha_8, \alpha_9, \alpha_{12}, \alpha_{16}\},$
- $\mathcal{A} = \{\alpha_4\}$ and
- $\mathcal{L} = \{\alpha_{13}\}.$

Using the method of §4.2, we take

- $Y = X_5X_6X_{10},$
- $X = X_1X_3X_7$ and then we have that
- $H = X_2YZ.$

In this case (4.2) is

$$s_5(a_8t_3 + a_{12}t_7) + s_6(-a_8t_1 - 2a_9t_3) + s_{10}(-a_{12}t_1 + 2a_{16}t_7) = 0.$$

For $a_{16} \neq a_9a_{12}^2/a_8^2$, we have $Y' = 1$ and Y is normal in \bar{H} . Further, $\bar{H}/Y = X_2X_{\mathcal{Z}}$, so as explained in Remark 4.4 we get the family of characters

$$\text{Irr}(U)_{\mathfrak{C}^2}^1 = \{\chi_{b_2}^{a_8, a_9, a_{12}, a_{16}^*} \mid a_8, a_9, a_{12}, a_{16}^* \in \mathbb{F}_q^\times, a_{16}^* \neq a_9(a_{12}/a_8)^2, b_2 \in \mathbb{F}_q\} \subseteq \text{Irr}(U)_{\mathfrak{C}^2},$$

where

$$\chi_{b_2}^{a_8, a_9, a_{12}, a_{16}^*} = \text{Ind}^{A \cup \mathcal{I}} \text{Inf}_{K \cup \mathcal{J}} \lambda_{b_2}^{a_8, a_9, a_{12}, a_{16}^*},$$

and $\lambda_{b_2}^{a_8, a_9, a_{12}, a_{16}^*} \in \text{Irr}(\bar{H}/Y)$ is defined in the usual way. We have that $\text{Irr}(U)_{\mathfrak{C}^2}^1$ consists of $q(q-1)^3(q-2)$ characters of degree q^4 .

Now suppose $a_{16} = a_9a_{12}^2/a_8^2$. We have $X' = \{x_{1,3,7}(t) \mid t \in \mathbb{F}_q\}$ and $Y' = \{x_{5,6,10}(s) \mid s \in \mathbb{F}_q\}$, where

$$x_{1,3,7}(t) = x_1(2a_9a_{12}t)x_3(-a_8a_{12}t)x_7(a_8^2t) \quad \text{and} \quad x_{5,6,10}(s) = x_5(2a_9a_{12}s)x_6(a_8a_{12}s)x_{10}(-a_8^2s).$$

We can take any complement of Y' in Y and we choose $\tilde{Y} = X_5X_{10}$. Then we have $\tilde{H}/\tilde{Y} = X_2X'Y\tilde{Z}/\tilde{Y}$ and $Y' \subseteq Z(\tilde{H}/\tilde{Y})$. From now on, we denote by $\lambda^{a_8, a_9, a_{12}}$ the character $\lambda_{b_2}^{a_8, a_9, a_{12}, a_{16}^*}$ with $a_{16} = a_9a_{12}^2/a_8^2$.

A computation in \tilde{H}/\tilde{Y} gives

$$[x_2(s), x_{1,3,7}(t)] = x_{5,6,10}(-st).$$

Therefore, \tilde{H}/\tilde{Y} is the direct product of Z and $X_2X'Y/\tilde{Y}$. Further $X_2X'Y/\tilde{Y}$ is isomorphic to the three-dimensional group V_f for $f(s, t) = -st$ from §4.1.

We label the linear characters of $X_2X'Y/\tilde{Y}$ by $\chi_{b_2, b_{1,3,7}}$. By tensoring these characters with $\lambda^{a_8, a_9, a_{12}}$ and then applying $\text{Ind}_{\tilde{H}X_K}^U \text{Inf}_{\tilde{H}/\tilde{Y}}^{\tilde{H}X_K}$ we obtain the family of characters

$$\text{Irr}(U)_{\mathfrak{C}^2}^2 = \{\chi_{b_2, b_{1,3,7}}^{a_8, a_9, a_{12}} \mid a_8, a_9, a_{12} \in \mathbb{F}_q^\times, b_2, b_{1,3,7} \in \mathbb{F}_q\},$$

which consists of $q^2(q-1)^3$ characters of degree q^3 .

Let $a_{5,6,10} \in \mathbb{F}_q^\times$. We write $\lambda^{a_8, a_9, a_{12}, a_{5,6,10}}$ for the linear character of $Y'Z$ defined by extending $\lambda^{a_8, a_9, a_{12}}$ to Y' in the usual way. By applying $\text{Ind}_{X'YX_ZX_K}^U \text{Inf}_{YZ/\tilde{Y}}^{X'YX_ZX_K}$ to these linear characters we obtain the family of characters

$$\text{Irr}(U)_{\mathfrak{C}^2}^3 = \{\chi^{a_8, a_9, a_{12}, a_{5,6,10}} \mid a_8, a_9, a_{12}, a_{5,6,10} \in \mathbb{F}_q^\times\},$$

which consists of $(q-1)^4$ characters of degree q^4 .

We have $\text{Irr}(U)_{\mathfrak{C}^2} = \text{Irr}(U)_{\mathfrak{C}^2}^1 \cup \text{Irr}(U)_{\mathfrak{C}^2}^2 \cup \text{Irr}(U)_{\mathfrak{C}^2}^3$ and this gives all the irreducible characters corresponding to \mathfrak{C}^2 .

The nonabelian core \mathfrak{C}^3 . This core occurs for $\Sigma = \{\alpha_{14}, \alpha_{15}\}$, and we have

- $\mathcal{S} = \{\alpha_2, \alpha_4, \alpha_6, \alpha_7, \alpha_8, \alpha_{10}, \alpha_{11}, \alpha_{14}, \alpha_{15}\}$,
- $\mathcal{Z} = \{\alpha_{10}, \alpha_{14}, \alpha_{15}\}$,
- $\mathcal{A} = \{\alpha_1, \alpha_3, \alpha_5\}$ and
- $\mathcal{L} = \{\alpha_9, \alpha_{12}, \alpha_{13}\}$.

Using the method of §4.2, we take

- $Y = X_6X_7X_{11}$,
- $X = X_2X_4X_8$ and then we have that
- $H = YZ$.

In this case (4.2) is

$$s_6(a_{10}t_4 + 2a_{14}t_8) + s_7(-a_{10}t_2 + a_{15}t_8) + s_{11}(-a_{14}t_2 + a_{15}t_4) = 0.$$

For $p \geq 5$, we have $Y' = 1$ and Y is normal in \tilde{H} . So as explained in Remark 4.4 we obtain

$$\text{Irr}(U)_{\mathfrak{C}^3}^{p \geq 5} = \{\chi^{a_{10}, a_{14}, a_{15}} \mid a_{10}, a_{14}, a_{15} \in \mathbb{F}_q^\times\}$$

by applying $\text{Ind}_{\mathcal{K} \cup \mathcal{J}}^{A \cup \mathcal{I}} \text{Inf}_{\mathcal{K} \cup \mathcal{J}}^{\mathcal{K} \cup \mathcal{J}}$ to the characters in $\text{Irr}(X_{\mathcal{S} \setminus (\mathcal{I} \cup \mathcal{J})})_{\mathcal{Z}}$. We have that $\text{Irr}(U)_{\mathfrak{C}^3}^{p \geq 5}$ consists of $(q-1)^3$ characters of degree q^6 .

Now suppose $p = 3$. We have $X' = \{x_{2,4,8}(t) \mid t \in \mathbb{F}_q\}$ and $Y' = \{x_{6,7,11}(s) \mid s \in \mathbb{F}_q\}$, where

$$x_{2,4,8}(t) = x_2(a_{15}t)x_4(a_{14}t)x_8(a_{10}t) \quad \text{and} \quad x_{6,7,11}(s) = x_6(a_{15}s)x_7(a_{14}s)x_{11}(-a_{10}s).$$

We can take $\tilde{Y} = X_6X_{11}$, and we have that $\tilde{H}/\tilde{Y} \cong X'Y'Z$ is abelian. This yields

$$\text{Irr}(U)_{\mathfrak{C}^3}^{p=3} = \{\chi_{b_{2,4,8}, b_{6,7,11}}^{a_{10}, a_{14}, a_{15}} \mid a_{10}, a_{14}, a_{15} \in \mathbb{F}_q^\times, b_{2,4,8}, b_{6,7,11} \in \mathbb{F}_q\},$$

where these characters are obtained by applying $\text{Ind}_{\tilde{H}X_K}^U \text{Inf}_{\tilde{H}/\tilde{Y}}^{\tilde{H}X_K}$ to the linear characters $\lambda_{b_{2,4,8}, b_{6,7,11}}^{a_{10}, a_{14}, a_{15}}$ of \tilde{H}/\tilde{Y} , which are labelled in the usual way. We have that $\text{Irr}(U)_{\mathfrak{C}^3}^{p=3}$ consists of $q^2(q-1)^3$ characters of degree q^5 .

The nonabelian core \mathfrak{C}^4 . This core occurs for $\Sigma = \{\alpha_{11}, \alpha_{12}, \alpha_{13}\}$, and we have

- $\mathcal{S} = \{\alpha_1, \dots, \alpha_{13}\}$,
- $\mathcal{Z} = \Sigma = \{\alpha_{11}, \alpha_{12}, \alpha_{13}\}$,

- $\mathcal{A} = \emptyset$ and
- $\mathcal{L} = \emptyset$.

Using the method of §4.2, we take

- $Y = X_5X_8X_9X_{10}$,
- $X = X_1X_3X_4X_7$ and then we have that
- $H = X_2X_6YZ$.

In this case (4.2) is

$$s_5(-a_{11}t_3^2 + a_{12}t_7) + s_8(-2a_{11}t_3 + a_{12}t_4) + s_9(-a_{11}t_1 + a_{13}t_4) + s_{10}(-a_{12}t_1 - a_{13}t_3) = 0.$$

For $p \geq 5$, we have that $Y' = 1$, and Y is normal in \bar{H} . Also we have $\bar{H}/Y \cong X_2X_6X_Z$ is abelian. For $b_2, b_6 \in \mathbb{F}_q$ we let $\lambda_{b_2, b_6}^{a_{11}, a_{12}, a_{13}} \in \text{Irr}(X_{S \setminus (\mathcal{I} \cup \mathcal{J})})_{\mathcal{Z}}$ be the linear character with the usual notation. Then as explained in Remark 4.4 we obtain

$$\text{Irr}(U)_{\mathfrak{C}^4}^{p \geq 5} = \{\chi_{b_2, b_6}^{a_{11}, a_{12}, a_{13}} \mid a_{11}, a_{12}, a_{13} \in \mathbb{F}_q^\times, b_2, b_6 \in \mathbb{F}_q\},$$

where $\chi_{b_2, b_6}^{a_{11}, a_{12}, a_{13}} = \text{Ind}^{\mathcal{A} \cup \mathcal{I}} \text{Inf}_{\mathcal{K} \cup \mathcal{J}} \lambda_{b_2, b_6}^{a_{11}, a_{12}, a_{13}}$. We have that $\text{Irr}(U)_{\mathfrak{C}^4}^{p \geq 5}$ is a family of $q^2(q-1)^3$ characters of degree q^4 .

Now suppose $p = 3$. We have $X' = \{x_{1,3,4,7}(t) \mid t \in \mathbb{F}_q\}$ and $Y' = \{x_{8,9,10}(s) \mid s \in \mathbb{F}_q\}$, where

$$x_{1,3,4,7}(t) = x_1(a_{13}t)x_3(-a_{12}t)x_4(a_{11}t)x_7(-a_{11}a_{12}t^2) \text{ and} \\ x_{8,9,10}(s) = x_8(a_{13}s)x_9(-a_{12}s)x_{10}(a_{11}s).$$

We can take $\tilde{Y} = X_5X_8X_9$, and we have $\tilde{H}/\tilde{Y} = X_2X_6X'YZ/\tilde{Y}$. By Lemma 4.3, we have $\text{Irr}(V \mid \lambda)$ is in bijection with $\text{Irr}(\tilde{H}/\tilde{Y} \mid \lambda)$.

We continue by considering $\text{Irr}(\tilde{H}/\tilde{Y} \mid \lambda)$ and note that Y' lies in the centre of \tilde{H}/\tilde{Y} . For $a_{8,9,10} \in \mathbb{F}_q^\times$, we let $\lambda^{a_{8,9,10}}$ be the extension of λ to $Y'Z$ with $\lambda^{a_{8,9,10}}(x_{8,9,10}(t)) = \phi(a_{8,9,10}t)$ for every $t \in \mathbb{F}_q$. Then $\text{Irr}(\tilde{H}/\tilde{Y} \mid \lambda)$ decomposes as the union of $\text{Irr}(\tilde{H}/\tilde{Y} \mid \lambda^{a_{8,9,10}})$ over $a_{8,9,10} \in \mathbb{F}_q^\times$ along with $\text{Irr}(\tilde{H}/Y \mid \lambda)$.

A computation in \tilde{H}/\tilde{Y} gives

$$[x_6(s), x_{1,3,4,7}(t)] = x_{8,9,10}(st).$$

Now by Lemma 2.1, we have that $\text{Irr}(\tilde{H}/\tilde{Y} \mid \lambda^{a_{8,9,10}})$ is in bijection with $\text{Irr}(X_2YZ/\tilde{Y} \mid \lambda^{a_{8,9,10}})$. Further, we have that $X_2YZ/\tilde{Y} \cong X_2Y'Z$ is abelian, and we label the linear characters in $\text{Irr}(X_2YZ/\tilde{Y} \mid \lambda^{a_{8,9,10}})$ as $\lambda_{b_2}^{a_{11}, a_{12}, a_{13}, a_{8,9,10}}$ in the usual way. This gives the family of characters

$$\text{Irr}(U)_{\mathfrak{C}^4}^{1, p=3} = \{\chi_{b_2}^{a_{11}, a_{12}, a_{13}, a_{8,9,10}} \mid a_{11}, a_{12}, a_{13}, a_{8,9,10} \in \mathbb{F}_q^\times, b_2 \in \mathbb{F}_q\},$$

where by Remark 4.4 we have $\chi_{b_2}^{a_{11}, a_{12}, a_{13}, a_{8,9,10}} = \text{Ind}_{X_2X'YX_ZX_K}^U \text{Inf}_{X_2YZ/\tilde{Y}}^{X_2X'YX_ZX_K} \lambda_{b_2}^{a_{11}, a_{12}, a_{13}, a_{8,9,10}}$.

We have that $\text{Irr}(U)_{\mathfrak{C}^4}^{1, p=3}$ consists of $q(q-1)^4$ irreducible characters of degree q^4 .

It remains to consider $\text{Irr}(\tilde{H}/Y \mid \lambda)$. We have $\tilde{H}/Y = X_2X'X_6YZ/Y$ and X_6 is central in \tilde{H}/Y . For $a_6 \in \mathbb{F}_q^\times$, we let $\mu^{a_6} : X_6Z \rightarrow \mathbb{F}_q$ be the extension of $\mu : Z \rightarrow \mathbb{F}_q$ to X_6 defined as usual, and $\lambda^{a_6} \in \text{Irr}(X_6Z)$ be such that $\lambda^{a_6} = \phi \circ \mu^{a_6}$. Then $\text{Irr}(\tilde{H}/Y \mid \lambda)$ decomposes as the union of $\text{Irr}(\tilde{H}/Y \mid \lambda^{a_6})$ over $a_6 \in \mathbb{F}_q^\times$ along with $\text{Irr}(\tilde{H}/X_6Y \mid \lambda)$.

A computation in \tilde{H}/Y gives

$$[x_2(t), x_{1,3,4,7}(s)] = x_6(-a_{12}st)x_{11}(a_{12}^2a_{13}s^3t).$$

We note that the quotient $\tilde{H}/(Y \ker \mu^{a_6}) = X_2 X' X_6 Y Z / (Y \ker \mu^{a_6})$ is isomorphic to the three-dimensional group V_f where $f(s, t) = a_{12}t(a_{11}a_{12}a_{13}s^3 - a_6s)$ is as given in §4.1, and we have that $\text{Irr}(\tilde{H}/Y \mid \lambda^{a_6})$ is in bijection with $\text{Irr}(\tilde{H}/(Y \ker \mu^{a_6}) \mid \lambda^{a_6})$. Thus we can apply the analysis of $\text{Irr}(V_f)$ in §4.1. We let $d = a_6/a_{11}a_{12}a_{13}$.

Suppose first that d is a square in \mathbb{F}_q . In this case we write $a_{1,6}$ for a_6 , and we define $e \in \mathbb{F}_q$ such that $e^2 = d$. We let

$$W' = \{x_{1,3,4,7}(es) \mid s \in \mathbb{F}_3\} \quad \text{and} \quad W_2 = \{x_2((e^{-3}/a_{11}a_{12}^2a_{13})t) \mid t \in \mathbb{F}_3\},$$

and we define $\lambda_{c_{1,3,4,7}, c_2}^{a_{11}, a_{12}, a_{13}, a_{1,6}}$ for $c_{1,3,4,7}, c_2 \in \mathbb{F}_3$ of $W_1 W_2 X_6 Y Z / (Y \ker \lambda^{a_{11}, a_{12}, a_{13}, a_{1,6}})$ as in §4.1. Then we get the family of characters

$$\text{Irr}(U)_{\mathfrak{C}^4}^{2,1,p=3} = \{\chi_{c_{1,3,4,7}, c_2}^{a_{11}, a_{12}, a_{13}, a_{1,6}} \mid a_{11}, a_{12}, a_{13} \in \mathbb{F}_q^\times, a_{1,6} \in a_{11}a_{12}a_{13}S_q, c_{1,3,4,7}, c_2 \in \mathbb{F}_3\},$$

where

$$\chi_{c_{1,3,4,7}, c_2}^{a_{11}, a_{12}, a_{13}, a_{1,6}} = \text{Ind}_{X'W_2X_6YX_ZX_K}^U \text{Inf}_{W_1W_2X_6YZ/(Y \ker \lambda^{a_{11}, a_{12}, a_{13}, a_{1,6}})}^{X'W_2X_6YX_ZX_K} \lambda_{c_{1,3,4,7}, c_2}^{a_{11}, a_{12}, a_{13}, a_{1,6}}$$

and S_q denotes the set of nonzero squares in \mathbb{F}_q . We have that $\text{Irr}(U)_{\mathfrak{C}^4}^{2,1,p=3}$ consists of $9(q-1)^4/2$ characters of degree $q^4/3$.

Suppose now that d is a nonsquare in \mathbb{F}_q . In this case we write $a_{2,6}$ for a_6 . We write $\lambda^{a_{11}, a_{12}, a_{13}, a_{2,6}}$ for the linear characters of $X_6 Y Z / (Y \ker \lambda^{a_{11}, a_{12}, a_{13}, a_{2,6}})$ in the usual notation. Then we get the family of characters

$$\text{Irr}(U)_{\mathfrak{C}^4}^{2,2,p=3} = \{\chi^{a_{11}, a_{12}, a_{13}, a_{2,6}} \mid a_{2,6} \in \mathbb{F}_q^\times \setminus (a_{11}a_{12}a_{13}S_q), a_{11}, a_{12}, a_{13} \in \mathbb{F}_q^\times\},$$

where

$$\chi^{a_{11}, a_{12}, a_{13}, a_{2,6}} = \text{Ind}_{X'X_6YX_ZX_K}^U \text{Inf}_{X_6YZ/(Y \ker \lambda^{a_{11}, a_{12}, a_{13}, a_{2,6}})}^{X'X_6YX_ZX_K} \lambda^{a_{11}, a_{12}, a_{13}, a_{2,6}}.$$

We have that $\text{Irr}(U)_{\mathfrak{C}^4}^{2,2,p=3}$ consists of $(q-1)^4/2$ characters of degree q^4 .

Similarly, we can analyse $\text{Irr}(\tilde{H}/X_6Y \mid \lambda)$ using the arguments for the three-dimensional group V_f where $f(s, t) = a_{11}a_{12}^2a_{13}s^3t$. Therefore, we get the family of characters

$$\text{Irr}(U)_{\mathfrak{C}^4}^{3,p=3} = \{\chi^{a_{11}, a_{12}, a_{13}} \mid a_{11}, a_{12}, a_{13} \in \mathbb{F}_q^\times\},$$

where the characters are given by

$$\chi^{a_{11}, a_{12}, a_{13}} = \text{Ind}_{X'X_6YX_ZX_K}^U \text{Inf}_{X_6YZ/X_6Y}^{X'X_6YX_ZX_K} \lambda^{a_{11}, a_{12}, a_{13}}.$$

We have that $\text{Irr}(U)_{\mathfrak{C}^4}^{3,p=3}$ consists of $(q-1)^3$ characters of degree q^4 .

Putting this together we obtain

$$\text{Irr}(U)_{\mathfrak{C}^4}^{p=3} = \text{Irr}(U)_{\mathfrak{C}^4}^{1,p=3} \cup \text{Irr}(U)_{\mathfrak{C}^4}^{2,1,p=3} \cup \text{Irr}(U)_{\mathfrak{C}^4}^{2,2,p=3} \cup \text{Irr}(U)_{\mathfrak{C}^4}^{3,p=3}.$$

The nonabelian core \mathfrak{C}^5 . This core occurs for $\Sigma = \{\alpha_{12}, \alpha_{13}, \alpha_{14}\}$, and we have

- $\mathcal{S} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{12}, \alpha_{13}, \alpha_{14}\},$
- $\mathcal{Z} = \Sigma = \{\alpha_{12}, \alpha_{13}, \alpha_{14}\},$
- $\mathcal{A} = \{\alpha_2\}$ and
- $\mathcal{L} = \{\alpha_{11}\}.$

Using the method of §4.2, we take

- $Y = X_1 X_7 X_8 X_9,$
- $X = X_4 X_5 X_6 X_{10}$ and then we have that
- $H = X_3 Y Z.$

In this case (4.2) is

$$s_1(-a_{14}t_4^2 + a_{12}t_{10}) + s_7(-a_{12}t_5 + a_{13}t_6) + s_8(a_{12}t_4 - 2a_{14}t_6) + s_9(a_{13}t_4 + a_{14}t_5) = 0.$$

For $p \geq 5$, we have that $Y' = 1$, and Y is normal in \bar{H} . Also we have $\bar{H}/Y \cong X_3X_Z$ is abelian. For $b_3 \in \mathbb{F}_q$ we let $\lambda_{b_3}^{a_{12}, a_{13}, a_{14}} \in \text{Irr}(X_{\mathcal{S} \setminus (\mathcal{I} \cup \mathcal{J})})_{\mathcal{Z}}$ be the linear character with the usual notation. Then as explained in Remark 4.4 we obtain

$$\text{Irr}(U)_{\mathfrak{e}^5}^{p \geq 5} = \{\chi_{b_3}^{a_{12}, a_{13}, a_{14}} \mid a_{12}, a_{13}, a_{14} \in \mathbb{F}_q^\times, b_3 \in \mathbb{F}_q\},$$

where $\chi_{b_3}^{a_{12}, a_{13}, a_{14}} = \text{Ind}_{\mathcal{K} \cup \mathcal{J}}^{\mathcal{A} \cup \mathcal{I}} \text{Inf}_{\mathcal{K} \cup \mathcal{J}} \lambda_{b_3}^{a_{12}, a_{13}, a_{14}}$. We have that $\text{Irr}(U)_{\mathfrak{e}^5}^{p \geq 5}$ is a family of $q(q-1)^3$ characters of degree q^5 .

Now suppose $p = 3$. We have $X' = \{x_{4,5,6,10}(t) \mid t \in \mathbb{F}_q\}$ and $Y' = \{x_{7,8,9}(s) \mid s \in \mathbb{F}_q\}$, where

$$x_{4,5,6,10} = x_4(-a_{14}t)x_5(a_{13}t)x_6(a_{12}t)x_{10}(a_{12}a_{14}t^2) \text{ and } x_{7,8,9} = x_7(a_{14}s)x_8(-a_{13}s)x_9(a_{12}s).$$

We can take $\tilde{Y} = X_1X_7X_8$, and we have $\tilde{H}/\tilde{Y} = X_3X'YZ/\tilde{Y}$. By Lemma 4.3, we have that $\text{Irr}(V \mid \lambda)$ is in bijection with $\text{Irr}(\tilde{H}/\tilde{Y} \mid \lambda)$.

A computation in \tilde{H}/\tilde{Y} gives

$$[x_3(s), x_{4,5,6,10}(t)] = x_{7,8,9}(-st).$$

We notice that \tilde{H}/\tilde{Y} is the direct product of Z and the 3-dimensional group $X_3X'Y/\tilde{Y} \cong X_3X'Y'$. Then the analysis in §4.1 applies with $f(s, t) = -st$.

We label the linear characters of $X_3X'Y/\tilde{Y}$ by $\chi_{b_3, b_{4,5,6,10}}^{a_{12}, a_{13}, a_{14}}$. By tensoring these characters with $\lambda^{a_{12}, a_{13}, a_{14}}$ and then applying $\text{Ind}_{H_{X_K}}^U \text{Inf}_{\tilde{H}/\tilde{Y}}^{\tilde{H}X_K}$ we obtain the family of characters

$$\text{Irr}(U)_{\mathfrak{e}^5}^{1, p=3} = \{\chi_{b_3, b_{4,5,6,10}}^{a_{12}, a_{13}, a_{14}} \mid a_{12}, a_{13}, a_{14} \in \mathbb{F}_q^\times, b_3, b_{4,5,6,10} \in \mathbb{F}_q\},$$

which consists of $q^2(q-1)^3$ characters of degree q^4 .

Let us fix $a_{7,8,9} \in \mathbb{F}_q^\times$. We write $\lambda^{a_{12}, a_{13}, a_{14}, a_{7,8,9}}$ for the linear character of $Y'Z$ defined in the usual way. By applying $\text{Ind}_{X'YX_ZX_K}^U \text{Inf}_{YZ/\tilde{Y}}^{X'YX_ZX_K}$ to these linear characters we obtain the family of characters

$$\text{Irr}(U)_{\mathfrak{e}^5}^{2, p=3} = \{\chi^{a_{12}, a_{13}, a_{14}, a_{7,8,9}} \mid a_{12}, a_{13}, a_{14}, a_{7,8,9} \in \mathbb{F}_q^\times\},$$

which consists of $(q-1)^4$ characters of degree q^5 .

We have $\text{Irr}(U)_{\mathfrak{e}^5}^{p=3} = \text{Irr}(U)_{\mathfrak{e}^5}^{1, p=3} \cup \text{Irr}(U)_{\mathfrak{e}^5}^{2, p=3}$.

APPENDIX: TABLES OF PARAMETRIZATION OF $\text{Irr}(U)$

This appendix contains a parametrization of the irreducible characters of U_{B_4} , U_{C_4} and U_{F_4} when p is not a very bad prime for U , that is $p \neq 2$.

The notation in the tables is as follows. The first column corresponds to the families of the form \mathcal{F}_Σ , where \mathcal{F}_Σ is the family of irreducible characters of U arising from an antichain Σ . The second column contains character labels for those families as explained in Sections 3 and 4. For a fixed core $(\mathcal{S}, \mathcal{Z}, \mathcal{A}, \mathcal{L}, \mathcal{K})$, we define

$$I_{\mathcal{A}} = \{i \in \{1, \dots, |\Phi^+|\} \mid \alpha_i \in \mathcal{A}\},$$

and define $I_{\mathcal{L}}$ similarly. In case of nonabelian cores, $I_{\mathcal{I}}$ and $I_{\mathcal{J}}$ are also defined in the same fashion. The third column contains $I_{\mathcal{A}}$ and $I_{\mathcal{L}}$. We note that \mathcal{K} can be determined from \mathcal{A} ,

\mathcal{L} and the labels of the characters. For the abelian cores, we recall that the irreducible characters in the family are obtained by applying $\text{Ind}^A \text{Inf}_{\mathcal{K}}$ to the linear characters in $\text{Irr}(X_S)_{\mathcal{Z}}$. We use the **bold** font to identify nonabelian cores. In these cases, we also use the second column to give any relation between the indices and the third column to give some information on the construction of these characters. In the case where we have $Y' = 1$ and Y is normal in \bar{H} , we give $I_{\mathcal{I}}$ and $I_{\mathcal{J}}$ in the third column, as in this case the irreducible characters are given by applying $\text{Ind}^{A \cup \mathcal{I}} \text{Inf}_{\mathcal{K} \cup \mathcal{J}}$ to linear characters in $\text{Irr}(X_{S \setminus (\mathcal{I} \cup \mathcal{J})})_{\mathcal{Z}}$. In other cases, we refer the reader to the relevant part of §4.3. Finally, the fourth column records the number of irreducible characters in a family corresponding to some character labels, and the fifth column records their degree.

Parametrization of the irreducible characters of U_{B_4}

\mathcal{F}	χ	I	Number	Degree
\mathcal{F}_{lin}	$\chi_{b_1, b_2, b_3, b_4}$		q^4	1
\mathcal{F}_5	χ^{a_5}	$I_A = \{1\}, I_C = \{2\},$	$q - 1$	q
\mathcal{F}_6	χ^{a_6}	$I_A = \{2\}, I_C = \{3\},$	$q - 1$	q
\mathcal{F}_7	χ^{a_7}	$I_A = \{3\}, I_C = \{4\},$	$q - 1$	q
\mathcal{F}_8	$\chi_{b_2}^{a_8}$	$I_A = \{1\}, I_C = \{5\},$	$q(q - 1)$	q
\mathcal{F}_9	$\chi_{b_2}^{a_9}$	$I_A = \{1, 3\}, I_C = \{5, 6\},$	$q(q - 1)$	q^2
\mathcal{F}_{10}	$\chi_{b_3}^{a_{10}}$	$I_A = \{2, 4\}, I_C = \{6, 7\},$	$q(q - 1)$	q^2
\mathcal{F}_{11}	$\chi_{b_2, b_5, b_6}^{a_{11}}$	$I_A = \{1, 3\}, I_C = \{8, 9\},$	$q^3(q - 1)$	q^2
\mathcal{F}_{12}	$\chi^{a_6, a_{12}}$	$I_A = \{1, 3, 4, 7\},$ $I_C = \{2, 5, 9, 10\},$	$(q - 1)^2$	q^4
	$\chi_{b_4, b_3}^{a_{12}}$	$I_A = \{1, 4, 7\}, I_C = \{5, 9, 10\},$	$q^2(q - 1)$	q^3
\mathcal{F}_{13}	$\chi_{b_1, b_3}^{a_{13}}$	$I_A = \{2, 5, 6\}, I_C = \{8, 9, 11\},$	$q^2(q - 1)$	q^3
\mathcal{F}_{14}	$\chi_{b_2, b_6, b_{10}}^{a_9, a_{14}}$	$I_A = \{1, 3, 4, 7\},$ $I_C = \{5, 8, 11, 12\},$	$q^3(q - 1)^2$	q^4
	$\chi_{b_5, b_{10}}^{a_6, a_{14}}$	$I_A = \{1, 3, 4, 7\},$ $I_C = \{2, 8, 11, 12\},$	$q^2(q - 1)^2$	q^4
	$\chi_{b_2, b_3, b_5, b_{10}}^{a_{14}}$	$I_A = \{1, 4, 7\}, I_C = \{8, 11, 12\},$	$q^4(q - 1)$	q^3
\mathcal{F}_{15}	$\chi_{b_3, b_6, b_7}^{a_{11}, a_{15}}$	$I_A = \{1, 2, 4, 5, 8\},$ $I_C = \{9, 10, 12, 13, 14\},$	$q^3(q - 1)^2$	q^5
	$\chi_{b_3, b_7}^{a_9, a_{15}}$	$I_A = \{1, 2, 4, 5, 8\},$ $I_C = \{6, 10, 12, 13, 14\},$	$q^2(q - 1)^2$	q^5
	$\chi_{b_1, b_3, b_6, b_7}^{a_{15}}$	$I_A = \{2, 4, 5, 8\},$ $I_C = \{10, 12, 13, 14\},$	$q^4(q - 1)$	q^4
\mathcal{F}_{16}	$\chi_{b_2, b_4}^{a_8, a_{16}}$	$I_A = \{1, 3, 6, 7, 9, 10\},$ $I_C = \{5, 11, 12, 13, 14, 15\},$	$q^2(q - 1)^2$	q^6
	$\chi_{b_4}^{a_5, a_{16}}$	$I_A = \{2, 3, 6, 7, 9, 10\},$ $I_C = \{1, 11, 12, 13, 14, 15\},$	$q(q - 1)^2$	q^6
	$\chi_{b_1, b_2, b_4}^{a_{16}}$	$I_A = \{3, 6, 7, 9, 10\},$ $I_C = \{11, 12, 13, 14, 15\},$	$q^3(q - 1)$	q^5
$\mathcal{F}_{1,6}$	χ^{a_1, a_6}	$I_A = \{2\}, I_C = \{3\},$	$(q - 1)^2$	q
$\mathcal{F}_{1,7}$	χ^{a_1, a_7}	$I_A = \{3\}, I_C = \{4\},$	$(q - 1)^2$	q
$\mathcal{F}_{1,10}$	$\chi_{b_3}^{a_1, a_{10}}$	$I_A = \{2, 4\}, I_C = \{6, 7\},$	$q(q - 1)^2$	q^2
$\mathcal{F}_{2,7}$	χ^{a_2, a_7}	$I_A = \{3\}, I_C = \{4\},$	$(q - 1)^2$	q
$\mathcal{F}_{3,5}$	χ^{a_3, a_5}	$I_A = \{1\}, I_C = \{2\},$	$(q - 1)^2$	q
$\mathcal{F}_{3,8}$	$\chi_{b_2}^{a_3, a_8}$	$I_A = \{1\}, I_C = \{5\},$	$q(q - 1)^2$	q
$\mathcal{F}_{4,5}$	χ^{a_4, a_5}	$I_A = \{1\}, I_C = \{2\},$	$(q - 1)^2$	q
$\mathcal{F}_{4,6}$	χ^{a_4, a_6}	$I_A = \{2\}, I_C = \{3\},$	$(q - 1)^2$	q
$\mathcal{F}_{4,8}$	$\chi_{b_2}^{a_4, a_8}$	$I_A = \{1\}, I_C = \{5\},$	$q(q - 1)^2$	q
$\mathcal{F}_{4,9}$	$\chi_{b_2}^{a_4, a_9}$	$I_A = \{1, 3\}, I_C = \{5, 6\},$	$q(q - 1)^2$	q^2
$\mathcal{F}_{4,11}$	$\chi_{b_2, b_5, b_6}^{a_4, a_{11}}$	$I_A = \{1, 3\}, I_C = \{8, 9\},$	$q^3(q - 1)^2$	q^2
$\mathcal{F}_{4,13}$	$\chi_{b_1, b_3}^{a_4, a_{13}}$	$I_A = \{2, 5, 6\}, I_C = \{8, 9, 11\},$	$q^2(q - 1)^2$	q^3

\mathcal{F}	χ	I	Number	Degree
$\mathcal{F}_{5,6}$	$\chi_{b_3}^{a_5, a_6}$	$I_A = \{1\}, I_L = \{2\},$	$q(q-1)^2$	q
$\mathcal{F}_{5,7}$	$\chi_{b_3}^{a_5, a_7}$	$I_A = \{1, 3\}, I_L = \{2, 4\},$	$(q-1)^2$	q^2
$\mathcal{F}_{5,10}$	$\chi_{b_1, b_3}^{a_5, a_{10}}$	$I_A = \{2, 4\}, I_L = \{6, 7\},$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{6,7}$	$\chi_{b_4}^{a_6, a_7}$	$I_A = \{2\}, I_L = \{3\},$	$q(q-1)^2$	q
$\mathcal{F}_{6,8}$	$\chi_{b_4}^{a_6, a_8}$	$I_A = \{1, 3\}, I_L = \{2, 5\},$	$(q-1)^2$	q^2
$\mathcal{F}_{7,8}$	$\chi_{b_2}^{a_7, a_8}$	$I_A = \{1, 4\}, I_L = \{3, 5\},$	$q(q-1)^2$	q^2
$\mathcal{F}_{7,9}$	$\chi_{b_2, b_4}^{a_7, a_9}$	$I_A = \{1, 3\}, I_L = \{5, 6\},$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{7,11}$	$\chi_{b_2, b_4, b_5, b_6}^{a_7, a_{11}}$	$I_A = \{1, 3\}, I_L = \{8, 9\},$	$q^4(q-1)^2$	q^2
$\mathcal{F}_{7,13}$	$\chi_{b_1}^{a_7, a_{13}}$	$I_A = \{2, 4, 5, 6\},$ $I_L = \{3, 8, 9, 11\},$	$q(q-1)^2$	q^4
$\mathcal{F}_{8,9}$	$\chi_{b_2}^{a_8, a_9}$	$I_A = \{1, 3\}, I_L = \{5, 6\},$	$q(q-1)^2$	q^2
$\mathcal{F}_{8,10}$	$\chi_{b_3}^{a_8, a_{10}}$	$I_A = \{1, 2, 4\}, I_L = \{5, 6, 7\},$	$q(q-1)^2$	q^3
$\mathcal{F}_{8,12}$	$\chi_{b_2, b_3}^{a_6, a_8, a_{12}}$	$I_A = \{1, 3, 4, 7\},$ $I_L = \{2, 5, 9, 10\},$	$(q-1)^3$	q^4
	$\chi_{b_2, b_3}^{a_8, a_{12}}$	$I_A = \{1, 4, 7\}, I_L = \{5, 9, 10\},$	$q^2(q-1)^2$	q^3
$\mathcal{F}_{9,10}$	$\chi_{b_4}^{a_9, a_{10}}$	$I_A = \{2, 3, 6\}, I_L = \{1, 5, 7\},$	$q(q-1)^2$	q^3
$\mathcal{F}_{10,11}$	$\chi_{b_5}^{a_{10}, a_{11}}$	$I_A = \{1, 2, 3, 4\},$ $I_L = \{6, 7, 8, 9\},$	$q(q-1)^2$	q^4
$\mathcal{F}_{10,13}$	$\chi_{b_1}^{a_7, a_{10}, a_{13}}$	$I_A = \{2, 4, 5, 6\},$ $I_L = \{3, 8, 9, 11\},$	$q(q-1)^3$	q^4
	$\chi_{b_1, b_3, b_4}^{a_{10}, a_{13}}$	$I_A = \{2, 5, 6\}, I_L = \{8, 9, 11\},$	$q^3(q-1)^2$	q^3
$\mathcal{F}_{11,12}$	$\chi_{b_2, b_6}^{a_{11}, a_{12}}$	$I_A = \{1, 3, 4, 7\},$ $I_L = \{5, 8, 9, 10\},$	$q^2(q-1)^2$	q^4
$\mathcal{F}_{12,13}$	$\chi_{b_3}^{a_{12}, a_{13}}$	$I_A = \{1, 2, 4, 5, 6\},$ $I_L = \{7, 8, 9, 10, 11\},$	$q(q-1)^2$	q^5
$\mathcal{F}_{13,14}$	$\chi_{b_3}^{a_{10}, a_{13}, a_{14}}$	$I_A = \{1, 5\}, I_L = \{2, 4, 8\},$ $I_L = \{9, 12\}, I_J = \{6, 7, 11\}.$	$q(q-1)^3$	q^5
	$\chi^{a_7, a_{13}, a_{14}}$	$I_A = \{1, 4, 5, 8, 11\},$ $I_L = \{2, 3, 6, 9, 12\},$	$(q-1)^3$	q^5
	$\chi_{b_3, b_4}^{a_{13}, a_{14}}$	$I_A = \{1, 5, 8, 11\},$ $I_L = \{2, 6, 9, 12\},$	$q^2(q-1)^2$	q^4
$\mathcal{F}_{1,2,7}$	$\chi_{b_3}^{a_1, a_2, a_7}$	$I_A = \{3\}, I_L = \{4\},$	$(q-1)^3$	q
$\mathcal{F}_{1,4,6}$	$\chi_{b_3}^{a_1, a_4, a_6}$	$I_A = \{2\}, I_L = \{3\},$	$(q-1)^3$	q
$\mathcal{F}_{1,6,7}$	$\chi_{b_4}^{a_1, a_6, a_7}$	$I_A = \{2\}, I_L = \{3\},$	$q(q-1)^3$	q
$\mathcal{F}_{3,4,5}$	$\chi_{b_4}^{a_3, a_4, a_5}$	$I_A = \{1\}, I_L = \{2\},$	$(q-1)^3$	q
$\mathcal{F}_{3,4,8}$	$\chi_{b_2}^{a_3, a_4, a_8}$	$I_A = \{1\}, I_L = \{5\},$	$q(q-1)^3$	q
$\mathcal{F}_{4,5,6}$	$\chi_{b_3}^{a_4, a_5, a_6}$	$I_A = \{1\}, I_L = \{2\},$	$q(q-1)^3$	q
$\mathcal{F}_{4,6,8}$	$\chi_{b_3}^{a_4, a_6, a_8}$	$I_A = \{1, 3\}, I_L = \{2, 5\},$	$(q-1)^3$	q^2
$\mathcal{F}_{4,8,9}$	$\chi_{b_2}^{a_4, a_8, a_9}$	$I_A = \{1, 3\}, I_L = \{5, 6\},$	$q(q-1)^3$	q^2
$\mathcal{F}_{5,6,7}$	$\chi_{b_4}^{a_5, a_6, a_7}$	$I_A = \{1, 3\}, I_L = \{2, 4\},$	$(q-1)^3$	q^2
$\mathcal{F}_{6,7,8}$	$\chi_{b_4}^{a_6, a_7, a_8}$	$I_A = \{1, 3\}, I_L = \{2, 5\},$	$q(q-1)^3$	q^2
$\mathcal{F}_{7,8,9}$	$\chi_{b_2, b_4}^{a_7, a_8, a_9}$	$I_A = \{1, 3\}, I_L = \{5, 6\},$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{8,9,10}$	$\chi_{b_4}^{a_8, a_9, a_{10}}$	$I_A = \{2, 5, 6\}, I_L = \{1, 3, 7\},$	$q(q-1)^3$	q^3

TABLE 5. The parametrization of the irreducible characters of $U_{B_4}(q)$, where $q = p^e$ and $p \geq 3$.

Parametrization of the irreducible characters of U_{C_4}

\mathcal{F}	χ	I	Number	Degree
\mathcal{F}_{lin}	$\chi_{b_1, b_2, b_3, b_4}$		q^4	1
\mathcal{F}_5	χ^{a_5}	$I_A = \{1\}, I_L = \{2\},$	$q-1$	q
\mathcal{F}_6	χ^{a_6}	$I_A = \{2\}, I_L = \{3\},$	$q-1$	q

\mathcal{F}	χ	I	Number	Degree
\mathcal{F}_7	χ^{a_7}	$I_A = \{3\}, I_C = \{4\},$	$q-1$	q
\mathcal{F}_8	$\chi_{b_1}^{a_8}$	$I_A = \{2\}, I_C = \{5\},$	$q(q-1)$	q
\mathcal{F}_9	$\chi_{b_2}^{a_9}$	$I_A = \{1, 3\}, I_C = \{5, 6\},$	$q(q-1)$	q^2
\mathcal{F}_{10}	$\chi_{b_3}^{a_{10}}$	$I_A = \{2, 4\}, I_C = \{6, 7\},$	$q(q-1)$	q^2
\mathcal{F}_{11}	$\chi_{b_1}^{a_{11}}$	$I_A = \{2, 3, 6\}, I_C = \{5, 8, 9\},$	$q(q-1)$	q^3
\mathcal{F}_{12}	$\chi^{a_6, a_{12}}$	$I_A = \{1, 3, 4, 7\},$ $I_C = \{2, 5, 9, 10\},$	$(q-1)^2$	q^4
	$\chi_{b_2, b_3}^{a_{12}}$	$I_A = \{1, 4, 7\}, I_C = \{5, 9, 10\},$	$q^2(q-1)$	q^3
\mathcal{F}_{13}	$\chi_{b_1}^{a_8, a_{13}}$	$I_A = \{2, 3, 6\}, I_C = \{5, 9, 11\},$	$q(q-1)^2$	q^3
	$\chi^{a_5, a_{13}}$	$I_A = \{2, 3, 6\}, I_C = \{1, 9, 11\},$	$(q-1)^2$	q^3
	$\chi_{b_1, b_2}^{a_{13}}$	$I_A = \{3, 6\}, I_C = \{9, 11\},$	$q^2(q-1)$	q^2
\mathcal{F}_{14}	$\chi_{b_3}^{a_9, a_{14}}$	$I_A = \{1, 2, 4, 5, 7\},$ $I_C = \{6, 8, 10, 11, 12\},$	$q(q-1)^2$	q^5
	$\chi_{b_1, b_3, b_6}^{a_{14}}$	$I_A = \{2, 4, 5, 7\},$ $I_C = \{8, 10, 11, 12\},$	$q^3(q-1)$	q^4
\mathcal{F}_{15}	$\chi_{b_1}^{a_8, a_{15}}$	$I_A = \{2, 3, 4, 6, 7, 10\},$ $I_C = \{5, 9, 11, 12, 13, 14\},$	$q(q-1)^2$	q^6
	$\chi^{a_5, a_{15}}$	$I_A = \{2, 3, 4, 6, 7, 10\},$ $I_C = \{1, 9, 11, 12, 13, 14\},$	$(q-1)^2$	q^6
	$\chi_{b_1, b_2}^{a_{15}}$	$I_A = \{3, 4, 6, 7, 10\},$ $I_C = \{9, 11, 12, 13, 14\},$	$q^2(q-1)$	q^5
\mathcal{F}_{16}	$\chi_{b_1}^{a_8, a_{13}, a_{16}}$	$I_A = \{2, 3, 4, 6, 7, 10\},$ $I_C = \{5, 9, 11, 12, 14, 15\},$	$q(q-1)^3$	q^6
	$\chi^{a_5, a_{13}, a_{16}}$	$I_A = \{2, 3, 4, 6, 7, 10\},$ $I_C = \{1, 9, 11, 12, 14, 15\},$	$(q-1)^3$	q^6
	$\chi_{b_1, b_2}^{a_{13}, a_{16}}$	$I_A = \{3, 4, 6, 7, 10\},$ $I_C = \{9, 11, 12, 14, 15\},$	$q^2(q-1)^2$	q^5
	$\chi_{b_1}^{a_{11}, a_{16}}$	$I_A = \{2, 3, 4, 6, 7, 10\},$ $I_C = \{5, 8, 9, 12, 14, 15\},$	$q(q-1)^2$	q^6
	$\chi_{b_3}^{a_8, a_9, a_{16}}$	$I_A = \{1, 4, 5, 7, 10\},$ $I_C = \{2, 6, 12, 14, 15\},$	$q(q-1)^3$	q^5
	$\chi_{b_1, b_3, b_6}^{a_8, a_{16}}$	$I_A = \{2, 4, 7, 10\},$ $I_C = \{5, 12, 14, 15\},$	$q^3(q-1)^2$	q^4
	$\chi_{b_2}^{a_9, a_{16}}$	$I_A = \{1, 3, 4, 7, 10\},$ $I_C = \{5, 6, 12, 14, 15\},$	$q(q-1)^2$	q^5
	$\chi_{b_1, b_5}^{a_6, a_{16}}$	$I_A = \{2, 4, 7, 10\},$ $I_C = \{3, 12, 14, 15\},$	$q^2(q-1)^2$	q^4
	$\chi_{b_3}^{a_5, a_{16}}$	$I_A = \{2, 4, 7, 10\},$ $I_C = \{1, 12, 14, 15\},$	$q(q-1)^2$	q^4
	$\chi_{b_1, b_2, b_3}^{a_{16}}$	$I_A = \{4, 7, 10\}, I_C = \{12, 14, 15\},$	$q^3(q-1)$	q^3
$\mathcal{F}_{1,6}$	χ^{a_1, a_6}	$I_A = \{2\}, I_C = \{3\},$	$(q-1)^2$	q
$\mathcal{F}_{1,7}$	χ^{a_1, a_7}	$I_A = \{3\}, I_C = \{4\},$	$(q-1)^2$	q
$\mathcal{F}_{1,10}$	$\chi_{b_3}^{a_1, a_{10}}$	$I_A = \{2, 4\}, I_C = \{6, 7\},$	$q(q-1)^2$	q^2
$\mathcal{F}_{2,7}$	χ^{a_2, a_7}	$I_A = \{3\}, I_C = \{4\},$	$(q-1)^2$	q
$\mathcal{F}_{3,5}$	χ^{a_3, a_5}	$I_A = \{1\}, I_C = \{2\},$	$(q-1)^2$	q
$\mathcal{F}_{3,8}$	$\chi_{b_1}^{a_3, a_8}$	$I_A = \{2\}, I_C = \{5\},$	$q(q-1)^2$	q
$\mathcal{F}_{4,5}$	χ^{a_4, a_5}	$I_A = \{1\}, I_C = \{2\},$	$(q-1)^2$	q
$\mathcal{F}_{4,6}$	χ^{a_4, a_6}	$I_A = \{2\}, I_C = \{3\},$	$(q-1)^2$	q
$\mathcal{F}_{4,8}$	$\chi_{b_1}^{a_4, a_8}$	$I_A = \{2\}, I_C = \{5\},$	$q(q-1)^2$	q
$\mathcal{F}_{4,9}$	$\chi_{b_2}^{a_4, a_9}$	$I_A = \{1, 3\}, I_C = \{5, 6\},$	$q(q-1)^2$	q^2
$\mathcal{F}_{4,11}$	$\chi_{b_1}^{a_4, a_{11}}$	$I_A = \{2, 3, 6\}, I_C = \{5, 8, 9\},$	$q(q-1)^2$	q^3
$\mathcal{F}_{4,13}$	$\chi_{b_1}^{a_4, a_8, a_{13}}$	$I_A = \{2, 3, 6\}, I_C = \{5, 9, 11\},$	$q(q-1)^3$	q^3
	$\chi^{a_4, a_5, a_{13}}$	$I_A = \{2, 3, 6\}, I_C = \{1, 9, 11\},$	$(q-1)^3$	q^3
	$\chi_{b_1, b_2}^{a_4, a_{13}}$	$I_A = \{3, 6\}, I_C = \{9, 11\},$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{5,6}$	$\chi_{b_3}^{a_5, a_6}$	$I_A = \{1\}, I_C = \{2\},$	$q(q-1)^2$	q
$\mathcal{F}_{5,7}$	χ^{a_5, a_7}	$I_A = \{1, 3\}, I_C = \{2, 4\},$	$(q-1)^2$	q^2
$\mathcal{F}_{5,10}$	$\chi_{b_1, b_3}^{a_5, a_{10}}$	$I_A = \{2, 4\}, I_C = \{6, 7\},$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{6,7}$	$\chi_{b_4}^{a_6, a_7}$	$I_A = \{2\}, I_C = \{3\},$	$q(q-1)^2$	q

\mathcal{F}	χ	I	Number	Degree
$\mathcal{F}_{6,8}$	$\chi_{b_1, b_3}^{a_6, a_8}$	$I_A = \{2\}, I_C = \{5\},$	$q^2(q-1)^2$	q
$\mathcal{F}_{7,8}$	$\chi_{b_1}^{a_7, a_8}$	$I_A = \{2, 4\}, I_C = \{3, 5\},$	$q(q-1)^2$	q^2
$\mathcal{F}_{7,9}$	$\chi_{b_2, b_4}^{a_7, a_9}$	$I_A = \{1, 3\}, I_C = \{5, 6\},$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{7,11}$	$\chi_{b_1, b_4}^{a_7, a_{11}}$	$I_A = \{2, 3, 6\}, I_C = \{5, 8, 9\},$	$q^2(q-1)^2$	q^3
$\mathcal{F}_{7,13}$	$\chi_{b_1, b_4}^{a_7, a_8, a_{13}}$	$I_A = \{2, 3, 6\}, I_C = \{5, 9, 11\},$	$q^2(q-1)^3$	q^3
	$\chi_{b_5, a_7, a_{13}}^{a_4}$	$I_A = \{2, 3, 6\}, I_C = \{1, 9, 11\},$	$q(q-1)^3$	q^3
	$\chi_{b_1, b_2, b_4}^{a_7, a_{13}}$	$I_A = \{3, 6\}, I_C = \{9, 11\},$	$q^3(q-1)^2$	q^2
$\mathcal{F}_{8,9}$	$\chi_{b_2}^{a_8, a_9}$	$I_A = \{1, 5\}, I_C = \{2, 6\},$	$q(q-1)^2$	q^2
$\mathcal{F}_{8,10}$	$\chi_{b_1, b_3, b_7}^{a_8, a_{10}}$	$I_A = \{2, 4\}, I_C = \{5, 6\},$	$q^3(q-1)^2$	q^2
$\mathcal{F}_{8,12}$	$\chi^{a_6, a_8, a_{12}}$	$I_A = \{1, 3, 4, 5\},$ $I_C = \{2, 7, 9, 10\},$	$(q-1)^3$	q^4
	$\chi_{b_2, b_3}^{a_8, a_{12}}$	$I_A = \{1, 4, 5\}, I_C = \{7, 9, 10\},$	$q^2(q-1)^2$	q^3
$\mathcal{F}_{9,10}$	$\chi_{b_4}^{a_9, a_{10}}$	$I_A = \{2, 3, 6\}, I_C = \{1, 5, 7\},$	$q(q-1)^2$	q^3
$\mathcal{F}_{10,11}$	$\chi_{b_1, b_4, b_7}^{a_{10}, a_{11}}$	$I_A = \{2, 3, 6\}, I_C = \{5, 8, 9\},$	$q^3(q-1)^2$	q^3
$\mathcal{F}_{10,13}$	$\chi_{b_1, b_4, b_5, b_8}^{a_{10}, a_{13}}$	$I_A = \{2, 3, 6\}, I_C = \{7, 9, 11\},$	$q^4(q-1)^2$	q^3
$\mathcal{F}_{11,12}$	$\chi_{b_4, b_7}^{a_{11}, a_{12}}$	$I_A = \{1, 3, 5, 9\},$ $I_C = \{2, 6, 8, 10\},$	$q^2(q-1)^2$	q^4
$\mathcal{F}_{12,13}$	$\chi_{b_2, b_4, b_8}^{a_{12}, a_{13}}$	$I_A = \{1, 3, 5, 9\},$ $I_C = \{6, 7, 10, 11\},$	$q^3(q-1)^2$	q^4
$\mathcal{F}_{13,14}$	$\chi_{b_1, b_4}^{a_{13}, a_{14}}$	$I_A = \{2, 5, 6, 8, 11\},$ $I_C = \{3, 7, 9, 10, 12\},$	$q^2(q-1)^2$	q^5
$\mathcal{F}_{1,2,7}$	χ^{a_1, a_2, a_7}	$I_A = \{3\}, I_C = \{4\},$	$(q-1)^3$	q
$\mathcal{F}_{1,4,6}$	χ^{a_1, a_4, a_6}	$I_A = \{2\}, I_C = \{3\},$	$(q-1)^3$	q
$\mathcal{F}_{1,6,7}$	$\chi_{b_4}^{a_1, a_6, a_7}$	$I_A = \{2\}, I_C = \{3\},$	$q(q-1)^3$	q
$\mathcal{F}_{3,4,5}$	χ^{a_3, a_4, a_5}	$I_A = \{1\}, I_C = \{2\},$	$(q-1)^3$	q
$\mathcal{F}_{3,4,8}$	$\chi_{b_1}^{a_3, a_4, a_8}$	$I_A = \{2\}, I_C = \{5\},$	$q(q-1)^3$	q
$\mathcal{F}_{4,5,6}$	$\chi_{b_2}^{a_4, a_5, a_6}$	$I_A = \{1\}, I_C = \{2\},$	$q(q-1)^3$	q
$\mathcal{F}_{4,6,8}$	$\chi_{b_1, b_3}^{a_4, a_6, a_8}$	$I_A = \{2\}, I_C = \{5\},$	$q^2(q-1)^3$	q
$\mathcal{F}_{4,8,9}$	$\chi_{b_2}^{a_4, a_8, a_9}$	$I_A = \{1, 5\}, I_C = \{2, 6\},$	$q(q-1)^3$	q^2
$\mathcal{F}_{5,6,7}$	χ^{a_5, a_6, a_7}	$I_A = \{1, 3\}, I_C = \{2, 4\},$	$(q-1)^3$	q^2
$\mathcal{F}_{6,7,8}$	$\chi_{b_1}^{a_6, a_7, a_8}$	$I_A = \{2, 4\}, I_C = \{3, 5\},$	$q(q-1)^3$	q^2
$\mathcal{F}_{7,8,9}$	χ^{a_7, a_8, a_9}	$I_A = \{1, 4, 5\}, I_C = \{2, 3, 6\},$	$(q-1)^3$	q^3
$\mathcal{F}_{8,9,10}$	$\chi_{b_4}^{a_8, a_9, a_{10}}$	$I_A = \{2, 3, 6\}, I_C = \{1, 5, 7\},$	$q(q-1)^3$	q^3

TABLE 6. The parametrization of the irreducible characters of $U_{C_4}(q)$, where $q = p^e$ and $p \geq 3$.

Parametrization of the irreducible characters of U_{F_4}

\mathcal{F}	χ	I	Number	Degree
\mathcal{F}_{lin}	$\chi_{b_1, b_2, b_3, b_4}$		q^4	1
\mathcal{F}_5	χ^{a_5}	$I_A = \{1\}, I_C = \{2\},$	$q-1$	q^1
\mathcal{F}_6	χ^{a_6}	$I_A = \{2\}, I_C = \{3\},$	$q-1$	q^1
\mathcal{F}_7	χ^{a_7}	$I_A = \{3\}, I_C = \{4\},$	$q-1$	q^1
\mathcal{F}_8	$\chi_{b_2}^{a_8}$	$I_A = \{1, 3\}, I_C = \{5, 6\},$	$q(q-1)$	q^2
\mathcal{F}_9	$\chi_{b_2}^{a_9}$	$I_A = \{3\}, I_C = \{6\},$	$q(q-1)$	q^1
\mathcal{F}_{10}	$\chi_{b_3}^{a_{10}}$	$I_A = \{2, 4\}, I_C = \{6, 7\},$	$q(q-1)$	q^2
\mathcal{F}_{11}	$\chi_{b_2, b_5, b_6}^{a_{11}}$	$I_A = \{1, 3\}, I_C = \{8, 9\},$	$q^3(q-1)$	q^2
\mathcal{F}_{12}	$\chi^{a_6, a_{12}}$	$I_A = \{1, 3, 4, 7\},$ $I_C = \{2, 5, 8, 10\},$	$(q-1)^2$	q^4
	$\chi_{b_2, b_3}^{a_{12}}$	$I_A = \{1, 4, 7\}, I_C = \{5, 8, 10\},$	$q^2(q-1)$	q^3
\mathcal{F}_{13}	$\chi_{b_3}^{a_{13}}$	$I_A = \{3, 4, 7\}, I_C = \{6, 9, 10\},$	$q(q-1)$	q^3
\mathcal{F}_{14}	$\chi_{b_1, b_3}^{a_{14}}$	$I_A = \{2, 5, 6\}, I_C = \{8, 9, 11\},$	$q^2(q-1)$	q^3

\mathcal{F}	χ	I	Number	Degree
\mathcal{F}_{15}	$\chi_{b_2, b_5, b_6, b_9, b_{10}}^{a_{15}}$	$I_A = \{1, 3, 4, 7\},$ $I_C = \{8, 11, 12, 13\},$	$q^5(q-1)$	q^4
\mathcal{F}_{16}	$\chi_{b_2}^{a_9, a_{16}}$ $\chi_{b_2}^{a_6, a_{16}}$ $\chi_{b_2, b_3}^{a_{16}}$	$I_A = \{3, 4, 7\}, I_C = \{6, 10, 13\},$ $I_A = \{3, 4, 7\}, I_C = \{2, 10, 13\},$ $I_A = \{4, 7\}, I_C = \{10, 13\},$	$q(q-1)^2$ $(q-1)^2$ $q^2(q-1)$	q^3 q^3 q^2
\mathcal{F}_{17}	$\chi_{b_3, b_7}^{a_{11}, a_{17}}$ $\chi_{b_1, b_3, b_7, b_9}^{a_{17}}$	$I_A = \{1, 2, 4, 5, 6, 8\},$ $I_C = \{9, 10, 12, 13, 14, 15\},$ $I_A = \{2, 4, 5, 6, 8\},$ $I_C = \{10, 12, 13, 14, 15\},$	$q^2(q-1)^2$ $q^4(q-1)$	q^6 q^5
\mathcal{F}_{18}	$\chi_{b_2, b_5, b_6, b_9, b_{10}, b_{13}}^{a_{11}, a_{18}}$ $\chi_{b_2, b_5, b_6, b_8, b_9}^{a_{13}, a_{18}}$ $\chi_{b_2, b_6, b_9, b_{10}}^{a_8, a_{18}}$ $\chi_{b_2, b_5, b_{10}}^{a_9, a_{18}}$ $\chi_{b_5, b_{10}}^{a_6, a_{18}}$ $\chi_{b_2, b_3, b_5, b_{10}}^{a_{18}}$	$I_A = \{1, 3, 4, 7\},$ $I_C = \{8, 12, 15, 16\},$ $I_A = \{1, 3, 4, 7\},$ $I_C = \{10, 12, 15, 16\},$ $I_A = \{1, 3, 4, 7\},$ $I_C = \{5, 12, 15, 16\},$ $I_A = \{1, 3, 4, 7\},$ $I_C = \{6, 12, 15, 16\},$ $I_A = \{1, 3, 4, 7\},$ $I_C = \{2, 12, 15, 16\},$ $I_A = \{1, 4, 7\}, I_C = \{12, 15, 16\},$	$q^6(q-1)^2$ $q^5(q-1)^2$ $q^4(q-1)^2$ $q^3(q-1)^2$ $q^2(q-1)^2$ $q^4(q-1)$	q^4 q^4 q^4 q^4 q^4 q^3
\mathcal{F}_{19}	$\chi_{b_4}^{a_5, a_{19}}$ $\chi_{b_1, b_2, b_4}^{a_{19}}$	$I_A = \{2, 3, 6, 7, 8, 9, 10\},$ $I_C = \{1, 11, 12, 13, 14, 15, 17\},$ $I_A = \{3, 6, 7, 8, 9, 10\},$ $I_C = \{11, 12, 13, 14, 15, 17\},$	$q(q-1)^2$ $q^3(q-1)$	q^7 q^6
\mathcal{F}_{20}	$\chi_{b_{11}}^{a_9, a_{14}, a_{15}, a_{20}}$ $\chi_{b_3, b_6, b_{11}}^{a_{14}, a_{15}, a_{20}}$ $\chi^{a_{11}, a_{13}, a_{14}, a_{20}}$ $\chi_{b_1, b_3, b_8, b_9}^{a_{13}, a_{14}, a_{20}}$ $\chi_{b_3, b_7}^{a_{11}, a_{14}, a_{20}}$ $\chi_{b_1, b_3, b_7, b_9}^{a_{14}, a_{20}}$ $\chi_{b_{11}}^{a_9, a_{15}, a_{20}}$ $\chi_{b_3, b_6, b_{11}}^{a_{15}, a_{20}}$ $\chi^{a_{11}, a_{13}, a_{20}}$ $\chi_{b_1, b_3, b_8, b_9}^{a_{13}, a_{20}}$ $\chi_{b_6, b_7}^{a_{11}, a_{20}}$ $\chi_{b_1, b_7, b_8}^{a_9, a_{20}}$ $\chi_{b_3, b_7}^{a_8, a_{20}}$ $\chi_{b_1, b_3, b_6, b_7}^{a_{20}}$	$I_A = \{1, 2, 3, 4, 5, 8, 12\},$ $I_C = \{6, 7, 10, 13, 16, 17, 18\},$ $I_A = \{1, 2, 4, 5, 8, 12\},$ $I_C = \{7, 10, 13, 16, 17, 18\},$ $I_A = \{1, 2, 3, 4, 5, 6, 10\},$ $I_C = \{7, 8, 9, 12, 16, 17, 18\},$ $I_A = \{2, 4, 5, 6, 10\},$ $I_C = \{7, 12, 16, 17, 18\},$ $I_A = \{1, 2, 4, 5, 8, 10\},$ $I_C = \{6, 9, 12, 16, 17, 18\},$ $I_A = \{2, 4, 5, 6, 10\},$ $I_C = \{8, 12, 16, 17, 18\},$ $I_A = \{1, 2, 3, 4, 5, 8, 12\},$ $I_C = \{6, 7, 10, 13, 16, 17, 18\},$ $I_A = \{1, 2, 4, 5, 8, 12\},$ $I_C = \{7, 10, 13, 16, 17, 18\},$ $I_A = \{1, 2, 3, 4, 5, 6, 10\},$ $I_C = \{7, 8, 9, 12, 16, 17, 18\},$ $I_A = \{2, 4, 5, 6, 10\},$ $I_C = \{7, 12, 16, 17, 18\},$ $I_A = \{1, 2, 3, 4, 5, 10\},$ $I_C = \{8, 9, 12, 16, 17, 18\},$ $I_A = \{2, 4, 5, 6, 10\},$ $I_C = \{3, 12, 16, 17, 18\},$ $I_A = \{1, 2, 4, 5, 10\},$ $I_C = \{6, 12, 16, 17, 18\},$ $I_A = \{2, 4, 5, 10\},$ $I_C = \{12, 16, 17, 18\},$	$q(q-1)^4$ $q^3(q-1)^3$ $(q-1)^4$ $q^4(q-1)^3$ $q^2(q-1)^3$ $q^4(q-1)^2$ $q(q-1)^3$ $q^3(q-1)^2$ $(q-1)^3$ $q^4(q-1)^2$ $q^2(q-1)^2$ $q^3(q-1)^2$ $q^2(q-1)^2$ $q^4(q-1)$	q^7 q^6 q^7 q^5 q^6 q^5 q^7 q^6 q^7 q^5 q^6 q^5 q^5 q^4
\mathcal{F}_{21}	$\chi_{b_1}^{a_{14}, a_{21}}$ $\chi_{b_2, b_5}^{a_{11}, a_{21}}$ $\chi_{b_9}^{a_5, a_{21}}$ $\chi_{b_1, b_2, b_9}^{a_{21}}$	$I_A = \{2, 3, 4, 5, 6, 7, 8, 10, 13\},$ $I_C = \{9, 11, 12, 15, 16, 17, 18, 19, 20\},$ $I_A = \{1, 3, 4, 6, 7, 8, 10, 13\},$ $I_C = \{9, 12, 15, 16, 17, 18, 19, 20\},$ $I_A = \{2, 3, 4, 6, 7, 8, 10, 13\},$ $I_C = \{1, 12, 15, 16, 17, 18, 19, 20\},$ $I_A = \{3, 4, 6, 7, 8, 10, 13\},$	$q(q-1)^2$ $q^2(q-1)^2$ $q(q-1)^2$ $q^3(q-1)$	q^9 q^8 q^8 q^7

\mathcal{F}	χ	I	Number	Degree
\mathcal{F}_{22}	$\chi_{b_1}^{a_{14}, a_{20}, a_{22}}$	$I_{\mathcal{C}} = \{12, 15, 16, 17, 18, 19, 20\},$ $I_{\mathcal{A}} = \{3, 4, 6, 7, 10, 13\}, I_{\mathcal{T}} = \{2, 9, 16\}.$ $I_{\mathcal{C}} = \{8, 12, 15, 17, 19, 21\}, I_{\mathcal{J}} = \{5, 11, 18\},$	$q(q-1)^3$	q^9
	$\chi_{b_2, b_5}^{a_8, a_{20}, a_{22}}$	$I_{\mathcal{A}} = \{1, 3, 4, 7, 10, 13, 16, 18\},$ $I_{\mathcal{C}} = \{6, 9, 11, 12, 15, 17, 19, 21\},$	$q^2(q-1)^3$	q^8
	$\chi_{b_6}^{a_5, a_{20}, a_{22}}$	$I_{\mathcal{A}} = \{2, 3, 4, 7, 10, 13, 16, 18\},$ $I_{\mathcal{C}} = \{1, 9, 11, 12, 15, 17, 19, 21\},$	$q(q-1)^3$	q^8
	$\chi_{b_1, b_2, b_6}^{a_{20}, a_{22}}$	$I_{\mathcal{A}} = \{3, 4, 7, 10, 13, 16, 18\},$ $I_{\mathcal{C}} = \{9, 11, 12, 15, 17, 19, 21\},$	$q^3(q-1)^2$	q^7
	$\chi_{b_1}^{a_{17}, a_{22}}$	$I_{\mathcal{A}} = \{2, 3, 4, 6, 7, 9, 10, 11, 13\},$ $I_{\mathcal{C}} = \{5, 8, 12, 14, 15, 16, 18, 19, 21\},$	$q(q-1)^2$	q^9
	$\chi_{b_2, b_4, b_5}^{a_{12}, a_{14}, a_{22}}$	$I_{\mathcal{A}} = \{1, 3, 7, 8, 9, 11, 13\},$ $I_{\mathcal{C}} = \{6, 10, 15, 16, 18, 19, 21\},$	$q^3(q-1)^3$	q^7
	$\chi_{b_4, b_{10}}^{a_5, a_{14}, a_{22}}$	$I_{\mathcal{A}} = \{2, 3, 6, 7, 9, 11, 13\},$ $I_{\mathcal{C}} = \{1, 8, 15, 16, 18, 19, 21\},$	$q^2(q-1)^3$	q^7
	$\chi_{b_1, b_2, b_4, b_{10}}^{a_{14}, a_{22}}$	$I_{\mathcal{A}} = \{3, 6, 7, 9, 11, 13\},$ $I_{\mathcal{C}} = \{8, 15, 16, 18, 19, 21\},$	$q^4(q-1)^2$	q^6
	$\chi_{b_2, b_5, b_6}^{a_{12}, a_{22}}$	$I_{\mathcal{A}} = \{1, 3, 4, 7, 9, 11, 13\},$ $I_{\mathcal{C}} = \{8, 10, 15, 16, 18, 19, 21\},$	$q^3(q-1)^2$	q^7
	$\chi_{b_8}^{a_5, a_{10}, a_{22}}$	$I_{\mathcal{A}} = \{2, 3, 6, 7, 9, 11, 13\},$ $I_{\mathcal{C}} = \{1, 4, 15, 16, 18, 19, 21\},$	$q(q-1)^3$	q^7
	$\chi_{b_1, b_2, b_8}^{a_{10}, a_{22}}$	$I_{\mathcal{A}} = \{3, 6, 7, 9, 11, 13\},$ $I_{\mathcal{C}} = \{4, 15, 16, 18, 19, 21\},$	$q^3(q-1)^2$	q^6
	$\chi_{b_2, b_4, b_5}^{a_8, a_{22}}$	$I_{\mathcal{A}} = \{1, 3, 7, 9, 11, 13\},$ $I_{\mathcal{C}} = \{6, 15, 16, 18, 19, 21\},$	$q^3(q-1)^2$	q^6
	$\chi_{b_4, b_6}^{a_5, a_{22}}$	$I_{\mathcal{A}} = \{2, 3, 7, 9, 11, 13\},$ $I_{\mathcal{C}} = \{1, 15, 16, 18, 19, 21\},$	$q^2(q-1)^2$	q^6
	$\chi_{b_1, b_2, b_4, b_6}^{a_{22}}$	$I_{\mathcal{A}} = \{3, 7, 9, 11, 13\},$ $I_{\mathcal{C}} = \{15, 16, 18, 19, 21\},$	$q^4(q-1)$	q^5
\mathcal{F}_{23}	$\chi_{b_1, b_5}^{a_{11}, a_{18}, a_{23}}$	$I_{\mathcal{A}} = \{2, 3, 4, 6, 7, 9, 10, 13, 16\},$ $I_{\mathcal{C}} = \{8, 12, 14, 15, 17, 19, 20, 21, 22\},$	$q^2(q-1)^3$	q^9
	$\chi_{b_1}^{a_8, a_{18}, a_{23}}$	$I_{\mathcal{A}} = \{2, 3, 4, 6, 7, 9, 10, 13, 16\},$ $I_{\mathcal{C}} = \{5, 12, 14, 15, 17, 19, 20, 21, 22\},$	$q(q-1)^3$	q^9
	$\chi_{b_1, b_3, b_5}^{a_{18}, a_{23}}$	$I_{\mathcal{A}} = \{2, 4, 6, 7, 9, 10, 13, 16\},$ $I_{\mathcal{C}} = \{12, 14, 15, 17, 19, 20, 21, 22\},$	$q^3(q-1)^2$	q^8
	$\chi_{b_1, b_5}^{a_{15}, a_{23}}$	$I_{\mathcal{A}} = \{2, 3, 4, 6, 7, 9, 10, 13, 16\},$ $I_{\mathcal{C}} = \{8, 11, 12, 14, 17, 19, 20, 21, 22\},$	$q^2(q-1)^2$	q^9
	$\chi_{b_1, b_4}^{a_{11}, a_{12}, a_{23}}$	$I_{\mathcal{A}} = \{2, 5, 6, 8, 9, 10, 13, 16\},$ $I_{\mathcal{C}} = \{3, 7, 14, 17, 19, 20, 21, 22\},$	$q^2(q-1)^3$	q^8
	$\chi_{b_1, b_4, b_5, b_7}^{a_{11}, a_{23}}$	$I_{\mathcal{A}} = \{2, 3, 6, 9, 10, 13, 16\},$ $I_{\mathcal{C}} = \{8, 14, 17, 19, 20, 21, 22\},$	$q^4(q-1)^2$	q^7
	$\chi_{b_1, b_3}^{a_{12}, a_{23}}$	$I_{\mathcal{A}} = \{2, 4, 6, 7, 9, 10, 13, 16\},$ $I_{\mathcal{C}} = \{5, 8, 14, 17, 19, 20, 21, 22\},$	$q^2(q-1)^2$	q^8
	$\chi_{b_1, b_4, b_7}^{a_8, a_{23}}$	$I_{\mathcal{A}} = \{2, 3, 6, 9, 10, 13, 16\},$ $I_{\mathcal{C}} = \{5, 14, 17, 19, 20, 21, 22\},$	$q^3(q-1)^2$	q^7
	$\chi_{b_1, b_5}^{a_7, a_{23}}$	$I_{\mathcal{A}} = \{2, 4, 6, 9, 10, 13, 16\},$ $I_{\mathcal{C}} = \{3, 14, 17, 19, 20, 21, 22\},$	$q^2(q-1)^2$	q^7
	$\chi_{b_1, b_3, b_4, b_5}^{a_{23}}$	$I_{\mathcal{A}} = \{2, 6, 9, 10, 13, 16\},$ $I_{\mathcal{C}} = \{14, 17, 19, 20, 21, 22\},$	$q^4(q-1)$	q^6
\mathcal{F}_{24}	$\chi_{b_2}^{a_9, a_{16}, a_{24}}$	$I_{\mathcal{A}} = \{1, 3, 4, 5, 7, 8, 11, 12, 14, 15\},$ $I_{\mathcal{C}} = \{6, 10, 13, 17, 18, 19, 20, 21, 22, 23\},$	$q(q-1)^3$	q^{10}
	$\chi^{a_6, a_{16}, a_{24}}$	$I_{\mathcal{A}} = \{1, 3, 4, 5, 7, 8, 11, 12, 14, 15\},$ $I_{\mathcal{C}} = \{2, 10, 13, 17, 18, 19, 20, 21, 22, 23\},$	$(q-1)^3$	q^{10}
	$\chi_{b_2, b_3}^{a_{16}, a_{24}}$	$I_{\mathcal{A}} = \{1, 4, 5, 7, 8, 11, 12, 14, 15\},$ $I_{\mathcal{C}} = \{10, 13, 17, 18, 19, 20, 21, 22, 23\},$	$q^2(q-1)^2$	q^9
	$\chi_{b_2}^{a_{13}, a_{24}}$	$I_{\mathcal{A}} = \{1, 3, 4, 5, 7, 8, 11, 12, 14, 15\},$ $I_{\mathcal{C}} = \{6, 9, 10, 17, 18, 19, 20, 21, 22, 23\},$	$q(q-1)^2$	q^{10}
	$\chi_{b_3, b_9}^{a_{10}, a_{24}}$	$I_{\mathcal{A}} = \{1, 2, 5, 6, 8, 11, 12, 14, 15\},$ $I_{\mathcal{C}} = \{4, 7, 17, 18, 19, 20, 21, 22, 23\},$	$q^2(q-1)^2$	q^9

\mathcal{F}	χ	I	Number	Degree
	$\chi_{b_2, b_4, b_7}^{a_9, a_{24}}$	$I_A = \{1, 3, 5, 8, 11, 12, 14, 15\},$ $I_C = \{6, 17, 18, 19, 20, 21, 22, 23\},$	$q^3(q-1)^2$	q^8
	$\chi_{b_4, b_7}^{a_6, a_{24}}$	$I_A = \{1, 3, 5, 8, 11, 12, 14, 15\},$ $I_C = \{2, 17, 18, 19, 20, 21, 22, 23\},$	$q^2(q-1)^2$	q^8
	$\chi_{b_2}^{a_7, a_{24}}$	$I_A = \{1, 4, 5, 8, 11, 12, 14, 15\},$ $I_C = \{3, 17, 18, 19, 20, 21, 22, 23\},$	$q(q-1)^2$	q^8
	$\chi_{b_2, b_3, b_4}^{a_{24}}$	$I_A = \{1, 5, 8, 11, 12, 14, 15\},$ $I_C = \{17, 18, 19, 20, 21, 22, 23\},$	$q^3(q-1)$	q^7
$\mathcal{F}_{1,6}$	χ^{a_1, a_6}	$I_A = \{2\}, I_C = \{3\},$	$(q-1)^2$	q^1
$\mathcal{F}_{1,7}$	χ^{a_1, a_7}	$I_A = \{3\}, I_C = \{4\},$	$(q-1)^2$	q^1
$\mathcal{F}_{1,9}$	$\chi_{b_1}^{a_1, a_9}$	$I_A = \{3\}, I_C = \{6\},$	$q(q-1)^2$	q^1
$\mathcal{F}_{1,10}$	$\chi_{b_3}^{a_1, a_{10}}$	$I_A = \{2, 4\}, I_C = \{6, 7\},$	$q(q-1)^2$	q^2
$\mathcal{F}_{1,13}$	$\chi_{b_2}^{a_1, a_{13}}$	$I_A = \{3, 4, 7\}, I_C = \{6, 9, 10\},$	$q(q-1)^2$	q^3
$\mathcal{F}_{1,16}$	$\chi_{b_2}^{a_1, a_9, a_{16}}$	$I_A = \{3, 4, 7\}, I_C = \{6, 10, 13\},$	$q(q-1)^3$	q^3
	$\chi^{a_1, a_6, a_{16}}$	$I_A = \{3, 4, 7\}, I_C = \{2, 10, 13\},$	$(q-1)^3$	q^3
	$\chi_{b_2, b_3}^{a_1, a_{16}}$	$I_A = \{4, 7\}, I_C = \{10, 13\},$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{2,7}$	χ^{a_2, a_7}	$I_A = \{3\}, I_C = \{4\},$	$(q-1)^2$	q^1
$\mathcal{F}_{3,5}$	χ^{a_3, a_5}	$I_A = \{1\}, I_C = \{2\},$	$(q-1)^2$	q^1
$\mathcal{F}_{4,5}$	χ^{a_4, a_5}	$I_A = \{1\}, I_C = \{2\},$	$(q-1)^2$	q^1
$\mathcal{F}_{4,6}$	χ^{a_4, a_6}	$I_A = \{2\}, I_C = \{3\},$	$(q-1)^2$	q^1
$\mathcal{F}_{4,8}$	$\chi_{b_1}^{a_4, a_8}$	$I_A = \{1, 3\}, I_C = \{5, 6\},$	$q(q-1)^2$	q^2
$\mathcal{F}_{4,9}$	$\chi_{b_2}^{a_4, a_9}$	$I_A = \{3\}, I_C = \{6\},$	$q(q-1)^2$	q^1
$\mathcal{F}_{4,11}$	$\chi_{b_2, b_5, b_6}^{a_4, a_{11}}$	$I_A = \{1, 3\}, I_C = \{8, 9\},$	$q^3(q-1)^2$	q^2
$\mathcal{F}_{4,14}$	$\chi_{b_1, b_3}^{a_4, a_{14}}$	$I_A = \{2, 5, 6\}, I_C = \{8, 9, 11\},$	$q^2(q-1)^2$	q^3
$\mathcal{F}_{5,6}$	$\chi_{b_3}^{a_5, a_6}$	$I_A = \{1\}, I_C = \{2\},$	$q(q-1)^2$	q^1
$\mathcal{F}_{5,7}$	χ^{a_5, a_7}	$I_A = \{2, 4\}, I_C = \{1, 3\},$	$(q-1)^2$	q^2
$\mathcal{F}_{5,9}$	χ^{a_5, a_9}	$I_A = \{2, 3\}, I_C = \{1, 6\},$	$(q-1)^2$	q^2
$\mathcal{F}_{5,10}$	$\chi_{b_1, b_3}^{a_5, a_{10}}$	$I_A = \{2, 4\}, I_C = \{6, 7\},$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{5,13}$	$\chi^{a_5, a_{13}}$	$I_A = \{2, 3, 4, 7\},$ $I_C = \{1, 6, 9, 10\},$	$(q-1)^2$	q^4
$\mathcal{F}_{5,16}$	$\chi^{a_5, a_9, a_{16}}$	$I_A = \{2, 3, 4, 7\},$ $I_C = \{1, 6, 10, 13\},$	$(q-1)^3$	q^4
	$\chi_{b_3, b_6}^{a_5, a_{16}}$	$I_A = \{2, 4, 7\}, I_C = \{1, 10, 13\},$	$q^2(q-1)^2$	q^3
$\mathcal{F}_{6,7}$	$\chi_{b_4}^{a_6, a_7}$	$I_A = \{2\}, I_C = \{3\},$	$q(q-1)^2$	q^1
$\mathcal{F}_{7,8}$	$\chi_{b_2, b_4}^{a_7, a_8}$	$I_A = \{1, 3\}, I_C = \{5, 6\},$	$q^2(q-1)^2$	q^2
$\mathcal{F}_{7,9}$	$\chi_{b_2, b_4}^{a_7, a_9}$	$I_A = \{3\}, I_C = \{6\},$	$q^2(q-1)^2$	q^1
$\mathcal{F}_{7,11}$	$\chi_{b_2, b_4, b_5, b_6}^{a_7, a_{11}}$	$I_A = \{1, 3\}, I_C = \{8, 9\},$	$q^4(q-1)^2$	q^2
$\mathcal{F}_{7,14}$	$\chi_{b_1}^{a_7, a_{14}}$	$I_A = \{2, 4, 5, 6\},$ $I_C = \{3, 8, 9, 11\},$	$q(q-1)^2$	q^4
$\mathcal{F}_{8,9}$	$\chi_{b_2}^{a_8, a_9}$	$I_A = \{1, 3\}, I_C = \{5, 6\},$	$q(q-1)^2$	q^2
$\mathcal{F}_{8,10}$	$\chi_{b_4}^{a_8, a_{10}}$	$I_A = \{2, 3, 6\}, I_C = \{1, 5, 7\},$	$q(q-1)^2$	q^3
$\mathcal{F}_{8,13}$	$\chi^{a_5, a_8, a_{13}}$	$I_A = \{2, 3, 4, 6\},$ $I_C = \{1, 7, 9, 10\},$	$(q-1)^3$	q^4
	$\chi_{b_1, b_2}^{a_8, a_{13}}$	$I_A = \{3, 4, 6\}, I_C = \{7, 9, 10\},$	$q^2(q-1)^2$	q^3
$\mathcal{F}_{8,16}$	$\chi_{b_2, b_9}^{a_8, a_{16}}$	$I_A = \{1, 3, 4, 7\},$ $I_C = \{5, 6, 10, 13\},$	$q^2(q-1)^2$	q^4
$\mathcal{F}_{9,10}$	$\chi_{b_4}^{a_9, a_{10}}$	$I_A = \{2, 6\}, I_C = \{3, 7\},$	$q(q-1)^2$	q^2
$\mathcal{F}_{9,12}$	$\chi_{b_2}^{a_9, a_{12}}$	$I_A = \{1, 3, 4, 7\},$ $I_C = \{5, 6, 8, 10\},$	$q(q-1)^2$	q^4
$\mathcal{F}_{10,11}$	$\chi_{b_5}^{a_{10}, a_{11}}$	$I_A = \{1, 2, 3, 4\},$ $I_C = \{6, 7, 8, 9\},$	$q(q-1)^2$	q^4
$\mathcal{F}_{10,14}$	$\chi_{b_1}^{a_7, a_{10}, a_{14}}$	$I_A = \{2, 4, 5, 6\},$ $I_C = \{3, 8, 9, 11\},$	$q(q-1)^3$	q^4
	$\chi_{b_1, b_3, b_4}^{a_{10}, a_{14}}$	$I_A = \{2, 5, 6\}, I_C = \{8, 9, 11\},$	$q^3(q-1)^2$	q^3
$\mathcal{F}_{11,12}$	$\chi_{b_2}^{a_{10}, a_{11}, a_{12}}$	$I_A = \{1, 3, 4, 7\},$ $I_C = \{5, 6, 8, 9\},$	$q(q-1)^3$	q^4
	$\chi^{a_6, a_{11}, a_{12}}$	$I_A = \{1, 3, 5, 8\},$ $I_C = \{2, 4, 7, 9\},$	$(q-1)^3$	q^4

\mathcal{F}	χ	I	Number	Degree
	$\chi_{b_2, b_3}^{a_{11}, a_{12}}$	$I_A = \{1, 5, 8\}, I_C = \{4, 7, 9\},$	$q^2(q-1)^2$	q^3
$\mathcal{F}_{11,13}$	$\chi_{b_1, b_2}^{a_{5, a_{10}, a_{11}, a_{13}}}$ $(a_{11}^* \neq a_5 a_{13}^2 / a_{10}^2)$ $\chi_{b_1, 4, 7, b_2, 6, 9}^{a_{5, a_{10}, a_{13}}}$ $\chi_{a_{10}, a_{11}, a_{13}}^{a_{5, a_{11}, a_{13}}}$ $\chi_{b_1, b_2}^{a_{11}, a_{13}}$	$I_A = \{3\}, I_I = \{1, 4, 7\},$ $I_C = \{8\}, I_J = \{2, 6, 9\}.$ See \mathcal{C}^1 in §4.3 $I_A = \{2, 3, 6, 9\},$ $I_C = \{1, 4, 7, 8\},$ $I_A = \{2, 3, 7, 9\},$ $I_C = \{1, 4, 6, 8\},$ $I_A = \{3, 7, 9\}, I_C = \{4, 6, 8\},$	$(q-1)^3(q-2)$ $q^2(q-1)^3$ $(q-1)^3$ $(q-1)^3$ $q^2(q-1)^2$	q^4 q^3 q^4 q^4 q^3
$\mathcal{F}_{11,16}$	$\chi_{b_2, b_5, b_6}^{a_{11}, a_{16}}$	$I_A = \{1, 3, 4, 7\},$ $I_C = \{8, 9, 10, 13\},$	$q^3(q-1)^2$	q^4
$\mathcal{F}_{12,13}$	$\chi_{b_2}^{a_9, a_{12}, a_{13}}$ $\chi^{a_6, a_{12}, a_{13}}$ $\chi_{b_2, b_3}^{a_{12}, a_{13}}$	$I_A = \{3, 4, 7, 10\},$ $I_C = \{1, 5, 6, 8\},$ $I_A = \{3, 4, 7, 10\},$ $I_C = \{1, 2, 5, 8\},$ $I_A = \{4, 7, 10\}, I_C = \{1, 5, 8\}.$	$q(q-1)^3$ $(q-1)^3$ $q^2(q-1)^2$	q^4 q^4 q^3
$\mathcal{F}_{12,14}$	$\chi^{a_7, a_{12}, a_{14}}$ $\chi_{b_3, b_4}^{a_{12}, a_{14}}$	$I_A = \{1, 2, 4, 5, 8\},$ $I_C = \{3, 6, 9, 10, 11\},$ $I_A = \{1, 2, 5, 8\},$ $I_C = \{6, 9, 10, 11\},$	$(q-1)^3$ $q^2(q-1)^2$	q^5 q^4
$\mathcal{F}_{12,16}$	$\chi_{b_2}^{a_8, a_9, a_{12}, a_{16}^*}$ $(a_{16}^* \neq a_9 a_{12}^2 / a_8^2)$ $\chi_{b_2, b_3, 7}^{a_8, a_9, a_{12}, a_{16}^*}$ $\chi_{b_2}^{a_8, a_{12}, a_{16}}$ $\chi_{b_2}^{a_9, a_{12}, a_{16}}$ $\chi^{a_6, a_{12}, a_{16}}$ $\chi_{b_2, b_3}^{a_{12}, a_{16}}$	$I_A = \{4\}, I_I = \{1, 3, 7\}$ $I_C = \{13\}, I_J = \{5, 6, 10\}$ See \mathcal{C}^2 in §4.3 See \mathcal{C}^2 in §4.3 $I_A = \{1, 3, 4, 7\},$ $I_C = \{5, 6, 10, 13\},$ $I_A = \{1, 3, 4, 7\},$ $I_C = \{5, 6, 10, 13\},$ $I_A = \{1, 3, 4, 7\},$ $I_C = \{2, 5, 10, 13\},$ $I_A = \{1, 4, 7\}, I_C = \{5, 10, 13\},$	$q(q-1)^3(q-2)$ $(q-1)^4$ $q^2(q-1)^3$ $q(q-1)^3$ $q(q-1)^3$ $(q-1)^3$ $q^2(q-1)^2$	q^4 q^4 q^3 q^4 q^4 q^3
$\mathcal{F}_{13,14}$	$\chi_{b_1, b_4, b_7}^{a_{13}, a_{14}}$	$I_A = \{2, 3, 6, 9\},$ $I_C = \{5, 8, 10, 11\},$	$q^3(q-1)^2$	q^4
$\mathcal{F}_{14,15}^{p \geq 5}$	$\chi^{a_{10}, a_{14}, a_{15}}$ $\chi_{b_4, b_7}^{a_{14}, a_{15}}$	$I_A = \{1, 3, 5\}, I_I = \{2, 4, 8\}$ $I_C = \{9, 12, 13\}, I_J = \{6, 7, 11\}$ $I_A = \{1, 3, 5, 8, 11\},$ $I_C = \{2, 6, 9, 12, 13\},$	$(q-1)^3$ $q^2(q-1)^2$	q^6 q^5
$\mathcal{F}_{14,15}^{p=3}$	$\chi_{b_2, 4, 8, b_6, 7, 11}^{a_{10}, a_{14}, a_{15}}$ $\chi_{b_4, b_7}^{a_{14}, a_{15}}$	See \mathcal{C}^3 in §4.3 $I_A = \{1, 3, 5, 8, 11\},$ $I_C = \{2, 6, 9, 12, 13\},$	$q^2(q-1)^3$ $q^2(q-1)^2$	q^5 q^5
$\mathcal{F}_{14,16}$	$\chi_{b_1, b_3}^{a_{14}, a_{16}}$	$I_A = \{2, 4, 5, 6, 7\},$ $I_C = \{8, 9, 10, 11, 13\},$	$q^2(q-1)^2$	q^5
$\mathcal{F}_{14,18}$	$\chi^{a_{13}, a_{14}, a_{18}}$ $\chi_{b_3, b_{10}}^{a_{14}, a_{18}}$	$I_A = \{1, 2, 3, 4, 5, 6, 7\},$ $I_C = \{8, 9, 10, 11, 12, 15, 16\},$ $I_A = \{1, 2, 4, 5, 6, 7\},$ $I_C = \{8, 9, 11, 12, 15, 16\},$	$(q-1)^3$ $q^2(q-1)^2$	q^7 q^6
$\mathcal{F}_{15,16}$	$\chi_{b_2, b_5, b_6, b_9, b_{10}}^{a_{15}, a_{16}}$	$I_A = \{1, 3, 4, 7\},$ $I_C = \{8, 11, 12, 13\},$	$q^5(q-1)^2$	q^4
$\mathcal{F}_{16,17}$	$\chi^{a_{11}, a_{16}, a_{17}}$ $\chi_{b_1, b_3, b_8, b_9}^{a_{16}, a_{17}}$	$I_A = \{1, 2, 3, 4, 5, 6, 10\},$ $I_C = \{7, 8, 9, 12, 13, 14, 15\},$ $I_A = \{2, 4, 5, 6, 10\},$ $I_C = \{7, 12, 13, 14, 15\},$	$(q-1)^3$ $q^4(q-1)^2$	q^7 q^5
$\mathcal{F}_{16,19}$	$\chi_{b_4}^{a_5, a_{16}, a_{19}}$ $\chi_{b_1, b_2, b_4}^{a_{16}, a_{19}}$	$I_A = \{2, 3, 6, 7, 9, 10, 13\},$ $I_C = \{1, 8, 11, 12, 14, 15, 17\},$ $I_A = \{3, 6, 7, 9, 10, 13\},$ $I_C = \{8, 11, 12, 14, 15, 17\},$	$q(q-1)^3$ $q^3(q-1)^2$	q^7 q^6
$\mathcal{F}_{17,18}$	$\chi_{b_{11}}^{a_9, a_{17}, a_{18}}$	$I_A = \{1, 2, 3, 4, 5, 8, 12\},$ $I_C = \{6, 7, 10, 13, 14, 15, 16\},$	$q(q-1)^3$	q^7

\mathcal{F}	χ	I	Number	Degree
	$\chi_{b_3, b_6, b_{11}}^{a_{17}, a_{18}}$	$I_A = \{1, 2, 4, 5, 8, 12\},$ $I_C = \{7, 10, 13, 14, 15, 16\},$	$q^3(q-1)^2$	q^6
$\mathcal{F}_{18,19}$	$\chi_{b_2, b_4, b_5}^{a_{18}, a_{19}}$	$I_A = \{1, 3, 7, 8, 9, 11, 15\},$ $I_C = \{6, 10, 12, 13, 14, 16, 17\},$	$q^3(q-1)^2$	q^7
$\mathcal{F}_{19,20}$	$\chi_{b_1, b_4}^{a_{19}, a_{20}}$	$I_A = \{2, 5, 6, 8, 9, 10, 14, 17\},$ $I_C = \{3, 7, 11, 12, 13, 15, 16, 18\},$	$q^2(q-1)^2$	q^8
$\mathcal{F}_{1,2,7}$	χ^{a_1, a_2, a_7}	$I_A = \{3\}, I_C = \{4\},$	$(q-1)^3$	q^1
$\mathcal{F}_{1,4,6}$	χ^{a_1, a_4, a_6}	$I_A = \{2\}, I_C = \{3\},$	$(q-1)^3$	q^1
$\mathcal{F}_{1,4,9}$	$\chi_{b_2}^{a_1, a_4, a_9}$	$I_A = \{3\}, I_C = \{6\},$	$q(q-1)^3$	q^1
$\mathcal{F}_{1,6,7}$	$\chi_{b_4}^{a_1, a_6, a_7}$	$I_A = \{2\}, I_C = \{3\},$	$q(q-1)^3$	q^1
$\mathcal{F}_{1,7,9}$	$\chi_{b_2, b_4}^{a_1, a_7, a_9}$	$I_A = \{3\}, I_C = \{6\},$	$q^2(q-1)^3$	q^1
$\mathcal{F}_{1,9,10}$	$\chi_{b_4}^{a_1, a_9, a_{10}}$	$I_A = \{2, 6\}, I_C = \{3, 7\},$	$q(q-1)^3$	q^2
$\mathcal{F}_{3,4,5}$	χ^{a_3, a_4, a_5}	$I_A = \{1\}, I_C = \{2\},$	$(q-1)^3$	q^1
$\mathcal{F}_{4,5,6}$	$\chi_{b_3}^{a_4, a_5, a_6}$	$I_A = \{1\}, I_C = \{2\},$	$q(q-1)^3$	q^1
$\mathcal{F}_{4,5,9}$	χ^{a_4, a_5, a_9}	$I_A = \{2, 3\}, I_C = \{1, 6\},$	$(q-1)^3$	q^2
$\mathcal{F}_{4,8,9}$	$\chi_{b_2}^{a_4, a_8, a_9}$	$I_A = \{1, 3\}, I_C = \{5, 6\},$	$q(q-1)^3$	q^2
$\mathcal{F}_{5,6,7}$	χ^{a_5, a_6, a_7}	$I_A = \{1, 3\}, I_C = \{2, 4\},$	$(q-1)^3$	q^2
$\mathcal{F}_{5,7,9}$	$\chi_{b_4}^{a_5, a_7, a_9}$	$I_A = \{2, 3\}, I_C = \{1, 6\},$	$q(q-1)^3$	q^2
$\mathcal{F}_{5,9,10}$	$\chi_{b_1, b_4}^{a_5, a_9, a_{10}}$	$I_A = \{2, 6\}, I_C = \{3, 7\},$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{7,8,9}$	$\chi_{b_2, b_4}^{a_7, a_8, a_9}$	$I_A = \{1, 3\}, I_C = \{5, 6\},$	$q^2(q-1)^3$	q^2
$\mathcal{F}_{8,9,10}$	$\chi_{b_4}^{a_8, a_9, a_{10}}$	$I_A = \{2, 3, 6\}, I_C = \{1, 5, 7\},$	$q(q-1)^3$	q^3
$\mathcal{F}_{11,12,13}^{p \geq 5}$	$\chi_{b_2, b_6}^{a_{11}, a_{12}, a_{13}}$	$I_I = \{1, 3, 4, 7\}$ $I_J = \{5, 8, 9, 10\}$	$q^2(q-1)^3$	q^4
$\mathcal{F}_{11,12,13}^{p=3}$	$\chi_{b_2}^{a_{11}, a_{12}, a_{13}, a_{18}, a_{19}, a_{20}}$ $\chi^{a_{11}, a_{12}, a_{13}, a_{2,6}}$ $\chi^{a_{11}, a_{12}, a_{13}}$ $\chi^{a_{11}, a_{12}, a_{13}, a_{1,6}}$ $\chi_{c_1, 3, 4, 7, c_2}$	See \mathfrak{C}^4 in §4.3 See \mathfrak{C}^4 in §4.3 See \mathfrak{C}^4 in §4.3 See \mathfrak{C}^4 in §4.3	$q(q-1)^4$ $(q-1)^4/2$ $(q-1)^3$ $9(q-1)^4/2$	q^4 q^4 q^4 $q^4/3$
$\mathcal{F}_{11,12,16}$	$\chi_{b_2, b_5, b_6}^{a_{11}, a_{12}, a_{16}}$	$I_A = \{1, 3, 4, 7\},$ $I_C = \{8, 9, 10, 13\},$	$q^3(q-1)^3$	q^4
$\mathcal{F}_{12,13,14}^{p \geq 5}$	$\chi_{b_3}^{a_{12}, a_{13}, a_{14}}$	$I_A = \{2\}, I_I = \{4, 5, 6, 10\}$ $I_C = \{11\}, I_J = \{1, 7, 8, 9\}$	$q(q-1)^3$	q^5
$\mathcal{F}_{12,13,14}^{p=3}$	$\chi^{a_{12}, a_{13}, a_{14}, a_{7,8,9}}$ $\chi_{b_3, b_4, 5, 6, 10}^{a_{12}, a_{13}, a_{14}}$	See \mathfrak{C}^5 in §4.3 See \mathfrak{C}^5 in §4.3	$(q-1)^4$ $q^2(q-1)^3$	q^5 q^4
$\mathcal{F}_{12,14,16}$	$\chi_{b_1, b_3}^{a_{12}, a_{14}, a_{16}}$	$I_A = \{2, 4, 5, 6, 10\},$ $I_C = \{7, 8, 9, 11, 13\},$	$q^2(q-1)^3$	q^5
$\mathcal{F}_{14,15,16}$	$\chi_{b_4}^{a_{14}, a_{15}, a_{16}}$	$I_A = \{3, 6, 7, 9, 11, 13\},$ $I_C = \{1, 2, 5, 8, 10, 12\},$	$q(q-1)^3$	q^6

TABLE 7. The parametrization of the irreducible characters of $U_{F_4}(q)$, where $q = p^e$ and $p \geq 3$.

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